



KOREA 2025 EARCOME9

The 9th ICMI-East Asia Regional Conference on
Mathematics Education

**Re: Visiting the Essence of Mathematics Education
in the Era of Digital Transformation**

JUL. 18 FRI ~ 22 TUE, 2025

Seoul National University, Siheung Campus

Proceeding Vol. 1

- Plenary Lectures
- Plenary Panel Discussions
- Invited Lectures

Co-Host



대한수학교육학회
The Korean Society of Educational Studies in Mathematics



한국수학교육학회
The Korean Society of Mathematical Education

KOREA 2025
EARCOME9

The 9th ICMI-East Asia Regional Conference on
Mathematics Education



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Welcome Message

It is with great pleasure that we welcome you to the 9th ICMI-East Asia Regional Conference on Mathematics Education (EARCOME 9), held from July 18 to 22, 2025, at Seoul National University's Siheung Campus in South Korea.

EARCOME has become a cornerstone for mathematics education in East Asia, bringing together researchers, educators, and practitioners from across the region and beyond. Its origin can be traced to the South East Asia Conferences on Mathematics Education (SEACME), which began in 1978. Since the first official EARCOME in 1998, the conference has evolved into a triennial gathering—except during ICME years—fostering collaboration, scholarly exchange, and innovation in mathematics education.

The theme for EARCOME 9, “RE: Visiting the Essence of Mathematics Education in the Era of Digital Transformation,” invites us to reflect deeply on foundational principles in mathematics education amid the profound shifts driven by digital technology. In this transformative age, questions of purpose, practice, and pedagogy are more urgent than ever.

As the International Program Committee Chair and Co-Chair, we are honored to support a program that reflects the diverse voices and vibrant scholarship across East Asia and beyond. This year's conference is co-hosted by The Korean Society of Educational Studies in Mathematics (KSESM) and The Korean Society of Mathematics Education (KSME). It includes an exceptional lineup of plenary lectures, invited presentations, topic study groups, and practice-oriented sessions for practitioners.

We extend our deepest gratitude to the Local Organizing Committee, co-chaired by Ho Kyoung Ko (Ajou University), GwiSoo Na (Cheongju National University of Education), and Jinho Kim (Daegu National University of Education), whose dedication and leadership have been instrumental in realizing this event. We also acknowledge the many LOC members and volunteers who have contributed their time and expertise to make EARCOME 9 a meaningful and enriching experience.

We are especially grateful for the generous support of the National Research Foundation of Korea (NRF), funded by the Ministry of Education (MOE), under Grant No. NRF-2025S1A8A4A01014320. Their support has been crucial in making this conference possible.

We warmly welcome you to EARCOME 9 and invite you to engage fully in the discussions, share your insights, and help shape a forward-looking, inclusive vision for mathematics education. We hope this conference will not only deepen your academic inquiry but also strengthen our shared commitment to building a collaborative and culturally grounded mathematics education agenda in East Asia.



Oh Nam Kwon

Chair, International Program Committee
Seoul National University, South Korea



Berinderjeet Kaur

Co-Chair, International Program Committee
National Institute of Education, Singapore

	July 18(Fri)	July 19(Sat)	July 20(Sun)
08:30~09:00		Networking	Networking
09:00~09:30		Plenary Panel Discussion 1 (CH) Coffee Break TSG 1 (#109) TSG 2-1 (#CHIA) TSG 2-2 (#CHIC) TSG 3 (#102) TSG 4-1 (#603) TSG 4-2 (#604) TSG 5 (#806) TSG 6-1 (#605) TSG 6-2 (#606) TSG 7 (#806-1) TSG 8 (#103) TSG 9 (#406) TSG 10 (#609)	Plenary Lecture 2 (CH) Coffee Break TSG 1 (#109) TSG 2-1 (#CHIA) TSG 2-2 (#CHIC) TSG 3 (#102) TSG 4-1 (#603) TSG 4-2 (#604) TSG 5 (#806) TSG 6-1 (#605) TSG 6-2 (#606) TSG 7 (#806-1) TSG 8 (#103) TSG 9 (#406) TSG 10 (#609)
09:30~10:00			
10:00~10:30			
10:30~11:00			
11:00~11:30			
11:30~12:00			
12:00~12:30			
12:30~13:00			
13:00~13:30	Registration Opening Ceremony (CH) Plenary Lecture 1 (CH)	Lunch/Networking	Lunch/Networking
13:30~14:00			
14:00~14:30		Invited Lecture 1-1 (CH) Invited Lecture 1-2 (#109) Invited Lecture 1-3 (#806) Invited Lecture 1-4 (#806-1)	Invited Lecture 2-1 (CH) Invited Lecture 2-2 (#109) Invited Lecture 2-3 (#806) Invited Lecture 2-4 (#806-1)
14:30~15:00		Poster Presentation 1 (1F Lobby)	Poster Presentation 2 (1F Lobby)
15:00~15:30		Coffee Break	Coffee Break
15:30~16:00		WG 1-1 (#109) WG 1-2 (#606) WG 1-3 (#605) WG 1-4 (#806-1) SSG 1-1 (CH) SSG 1-2 (#806) SSG 1-3 (#603) SSG 1-4 (#604)	WG 2-1 (#109) WG 2-2 (#806-1) WG 2-3 (#605) WG 2-4 (#606) SSG 2-1 (CH) SSG 2-2 (#806) SSG 2-3 (#603) SSG 2-4 (#604)
16:00~16:30			
16:30~17:00			
17:00~17:30			
17:30~18:00			
18:00~20:00	Welcome Reception (2F Lobby)		

* **TSG** | Topic Study Group
 * **WG** | Working Group
 * **SSG** | Special Sharing Group

* **CH** | Convention Hall

Program at a Glance

	July 21(Mon)												July 22(Tue)			
08:30~09:00	Networking												Networking			
09:00~09:30	Plenary Lecture 3 (CH)												E x h i b i t i o n	Invited Lecture 3-1 (CH)	Invited Lecture 3-2 (#109)	Invited Lecture 3-3 (#806)
09:30~10:00														Coffee Break		
10:00~10:30	TSG 1 (#109) TSG 2-1 (#CHIA) TSG 2-2 (#CHIC) TSG 3 (#102) TSG 4-1 (#603) TSG 4-2 (#604) TSG 5 (#806) TSG 6-1 (#605) TSG 6-2 (#606) TSG 7 (#806-1) TSG 8 (#103) TSG 9 (#406) TSG 10 (#609)													Plenary Lecture 4 (CH)		
10:30~11:00														Closing (CH)		
11:00~11:30																
11:30~12:00																
12:00~12:30																
12:30~13:00	Lunch/Networking IPC & AB Meeting (#102)															
13:00~13:30																
13:30~14:00																
14:00~14:30	Plenary Panel Discussion 2 (CH)															
14:30~15:00	Coffee Break															
15:00~15:30																
15:30~16:00	WG 3-1 (#109) WG 3-2 (#806-1) WG 3-3 (#605) WG 3-4 (#606) SSG 3-1 (CH) SSG 3-2 (#806)															
16:00~16:30																
16:30~17:00																
17:00~17:30																
17:30~18:00																
18:00~20:00	Conference Banquet															

Scientific Program

1. Plenary Lectures

Plenary Lecture 1

July 18, Friday

17:00~18:00

**STAYING THE COURSE OR GOING WITH THE FLOW:
ESSENCE OF MATHEMATICS EDUCATION IN EAST ASIA IN THE ERA OF DIGITAL TRANSFORMATION**
Moderator: Berinderjeet Kaur (Nanyang Technological University)
Speaker: Wee Tiong Seah (University of Melbourne)

Plenary Lecture 2

July 20, Sunday

09:00~10:00

**ADDRESSING ALIGNMENT OF COMPETENCY-BASED MATHEMATICS CURRICULUM:
CHALLENGES AND OPPORTUNITIES**
Moderator: Maitree Inprasitha (Khon Kaen University)
Speaker: Yoshinori Shimizu (University of Tsukuba)

Plenary Lecture 3

July 21, Monday

09:00~10:00

**BRIDGING ACHIEVEMENT-MOTIVATION GAPS:
A SYNERGISTIC APPROACH TO QUALITY MATHEMATICS EDUCATION**
Moderator: Fou-Lai Lin (National Taiwan Normal University)
Speaker: Kai-Lin Yang (National Taiwan Normal University)

Plenary Lecture 4

July 22, Tuesday

10:30~11:30

**A DESCRIPTIVE THEORY OF MATHEMATICS TEACHING:
WHAT CAN THE THEORY OF PRACTICAL RATIONALITY SAY ABOUT THE IMPROVEMENT OF
MATHEMATICS INSTRUCTION?**
Moderator: Oh Nam Kwon (Seoul National University)
Speaker: Patricio G. Herbst (University of Michigan)

2. Plenary Panel Discussions

Plenary Panel Discussion 1

Digital Transformation in Mathematics Classrooms: Rethinking Teaching and Learning in the AI Era

July 19, Saturday

09:00~10:00	Moderator: Lianghuo Fan (University of Macau)
	RETHINKING THE ROLE OF MATHEMATICS TEACHERS Yeap Ban Har (Pathlight School)
	RETHINKING MATHEMATICS EDUCATION IN THE AGE OF AI: TOWARD A DIGITALLY TRANSFORMED FUTURE Gabriele Kaiser (University of Hamburg)
	DIGITAL TRANSFORMATION IN MATHEMATICS EDUCATION: RETHINKING CLASSROOM PRACTICES WITH AI AND DIGITAL TECHNOLOGIES Frederick K.S. Leung (The University of Hong Kong)
	REIMAGINING MATHEMATICS EDUCATION THROUGH DIGITAL TRANSFORMATION JeongSuk Pang (Korea National University of Education)

Plenary Panel Discussion 2

The Evolution of Mathematics Education in East Asia

July 21, Monday

14:00~15:00	Moderator: Vilma Mesa (University of Michigan)
	MATHEMATICS EDUCATION IN SINGAPORE – AN INSIDER’S PERSPECTIVE Berinderjeet Kaur (Nanyang Technological University)
	KOREAN MATHEMATICS EDUCATION AT A CROSSROADS: BALANCING TRADITION, INNOVATION, AND EQUITY Oh Nam Kwon (Seoul National University)
	THE CHANGING LANDSCAPE OF MATHEMATICS EDUCATION IN THE PHILIPPINES: REFLECTIONS ON THREE DECADES OF ACTIVE ENGAGEMENT IN ASIA AND BEYOND Catherine Vistro-Yu (Ateneo de Manila University)
	PROGRESS IN MATHEMATICS EDUCATION IN JAPAN: A 30-YEAR RETROSPECTIVE AND FUTURE OUTLOOK Keiko Hino (Utsunomiya University)
	THE DEVELOPMENT OF MATHEMATICS EDUCATION IN HONG KONG: LOOKING BACK AND LOOKING FORWARD Ah Chee Ida Mok (The Chinese University of Hong Kong)

Scientific Program

3. Invited Lectures

Invited Lectures 1	
July 19, Saturday	
Invited Lecture 1-1	
14:00~15:00	RECENT DISCOURSE IN RESEARCH ON THE COMPETENCE OF MATHEMATICS TEACHERS Moderator: Norbert Noster (University of Würzburg) Speaker: Gabriele Kaiser (University of Hamburg)
Invited Lecture 1-2	
14:00~15:00	EAST ASIAN NARRATIVES OF MATHEMATICS IN AN ARTS-BASED RESEARCH PROJECT Moderator: Ban Heng Choy (Nanyang Technological University) Speaker: Caroline Yoon (University of Auckland)
Invited Lecture 1-3	
14:00~15:00	TRANSFORMING MATHEMATICS EDUCATION PRACTICES THROUGH INNOVATIVE PEDAGOGY AND DATA-DRIVEN PATHWAYS Moderator: Maitree Inprasitha (Khon Kaen University) Speaker: Padmanabhan Seshaiyer (George Mason University)
Invited Lecture 1-4	
14:00~15:00	TEACHING MATHEMATICS THROUGH APPLICATIONS: A CASE OF MATHEMATICS EDUCATION FOR NON-STEM UNDERGRADUATE STUDENTS Moderator: Masataka Koyama (Hiroshima University) Speaker: Mitsuru Kawazoe (Osaka Metropolitan University)
Invited Lectures 2	
July 20, Sunday	
Invited Lecture 2-1	
14:00~15:00	COMPUTATIONAL THINKING AS A CATALYST FOR CHANGE: REFLECTING ON WHAT, HOW, AND FOR WHOM WE TEACH MATHEMATICS IN THE DIGITAL AGE Moderator: Igor' Kontorovich (The University of Hong Kong) Speaker: Oi-Lam Ng (The Chinese University of Hong Kong)
Invited Lecture 2-2	
14:00~15:00	INTEGRATING AI AND DIGITAL TECHNOLOGIES IN STEM+I MATHEMATICS EDUCATION: OPPORTUNITIES AND CHALLENGES Moderator: Jihwa Noh (Pusan National University) Speaker: Ho Kyoung Ko (Ajou University)

Invited Lecture 2-3

14:00~15:00	EMPOWERING STUDENTS AS MATHEMATICIANS THROUGH COMMUNITY-BASED MATHEMATICAL MODELING Moderator: Hyunkyung Yoon (California Polytechnic University, Pomona) Speaker: Jennifer Suh (George Mason University)

Invited Lecture 2-4

14:00~15:00	TEACHING MATHEMATICS THROUGH PROBLEM POSING: A SYSTEMATIC REVIEW OF PROBLEM-POSING-BASED LEARNING RESEARCH Moderator: Yiming Cao (Beijing Normal University) Speaker: Jaepil Han & Jinfa Cai (University of Delaware)

Invited Lectures 3

July 22, Tuesday

Invited Lecture 3-1

09:00~10:00	THE IMPACT OF SELF-REGULATED LEARNING ON MATHEMATICS PERFORMANCE: INSIGHTS FROM EAST ASIAN STUDENTS IN PISA 2022 Moderator: JeongSuk Pang (Korea National University of Education) Speaker: Zhu Yan (East China Normal University)

Invited Lecture 3-2

09:00~10:00	CREATING MATHEMATICS CLASSROOMS WHERE ALL STUDENTS CAN SUCCEED AND ENJOY LEARNING Moderator: Jihwa Noh (Pusan National University) Speaker: Jinho Kim (Daegu National University of Education)

Invited Lecture 3-3

09:00~10:00	MATH IS FUN WITH BUNDLE-NUMBERS ON A BUNDLE-BUNDLE-BOARD Moderator: Kyungwon Lee (Seoul National University) Speaker: Allan Tarp (MATHeCADEMY.net)

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Plenary Lectures

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Plenary Lecture 1

**STAYING THE COURSE OR GOING WITH THE FLOW:
ESSENCE OF MATHEMATICS EDUCATION IN EAST ASIA IN THE ERA
OF DIGITAL TRANSFORMATION**

Wee Tiong Seah



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



STAYING THE COURSE OR GOING WITH THE FLOW: ESSENCE OF MATHEMATICS EDUCATION IN EAST ASIA IN THE ERA OF DIGITAL TRANSFORMATION

Wee Tiong Seah

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As the world enters the second quarter of the 21st Century, global megatrends include accelerating advances in digital technologies, geopolitical transition to multipolarity, and continuing threats to Earth's climate system. Thus countries need to develop their own human capital to their fullest potential, especially in mathematics and numeracy. For the six East Asian mathematics education systems in China, Hong Kong, Japan, Korea, Singapore, and Taiwan, it is likely that they will respond with: increased and refined focus on mathematics competencies, more overt emphasis on students' mathematics education well-being, boosting mathematical literacy (numeracy) within the community, and reinforcing the role of teachers as teacher. These suggest how pragmatic Confucianism underlies these responses. The discussion in this paper not only allows us to understand the enduring influence of Confucianism in mathematics education in the six East Asian jurisdictions, more importantly it is hoped that it also provides new perspectives and inspirations for other mathematics education systems (in East Asia and beyond) as they identify and build on cultural intersections.

Keywords: Competencies, Confucianism, MEW, numeracy, pragmatism, values, well-being

The world today: the era of digital transformation

At the time of writing this paper, the world was on the threshold of stepping into the second quarter of the 21st Century. This period in human history is arguably characterized by three significant megatrends, each with global impacts. These are, namely, the accelerating advances in digital technologies, the emergence of multipolarity in geopolitics, and the continuing threats to the Earth's climate system.

In the arena of technological advances, the pace of change has been nothing short of phenomenal. The exponential growth in computer power, for example, is vividly demonstrated through the Kurzweil Curve (Kurzweil, 1999). The next significant capability which can be handled by digital technologies is taking place quicker and sooner. What is perhaps even more significant in current times is that increasingly 'intelligent' digital technologies will soon be challenging the very notion of what humanity and being human is, not just in capability but also in form. Another aspect of digital technologies relates to social media. Amongst the different concerns associated with the prolific, idle and extended use of social media, emerging evidence of the erosion of some kinds of critical reasoning skills (e.g. Menichelli & Braccini, 2020) spells trouble for the human race, especially when these skills are needed more than ever before if we are to be able to interact with artificial intelligence (AI) applications more productively, and to be able to identify mis- and disinformation that are increasingly being pushed to the public by businesses and governments alike.

In geopolitics, an unipolar world order led by the United States after the Second World War has given way to multipolarity involving other nations such as China, India and Russia. These changes have implications for economies and have led to such phenomena as decoupling of technologies between China and the United States. How this new world order will look like is unclear, as the world

is currently (in 2025) experiencing a rather difficult transitional phase which in itself is fraught with uncertainties.

As for climate change, much has dominated the news globally over the last few decades, and the danger lies in what else we are not yet aware of. The world might only now be realizing that banning plastic use and replacing it with other recyclable materials can actually cause greater harm to the environment (The Economist, 2025). Creative solutions exist, but commercial and political interests often stand in the way of them being adopted. As another example, global efforts to slow down the deterioration of the world's climate system (and dwindling natural resources) might well be derailed by power-hungry and resource-hungry AI systems (Ren, 2023). To date, there is no viable solution through either producing energy-efficient AI systems or cooling relevant components of data centres more sustainably.

These significant megatrends are also inter-related to one another. As can be seen in the previous paragraphs, digital technologies play a key role across these scenarios. For this reason, I shall refer to this second quarter of the century as being an era of digital transformation, acknowledging that digital technologies are actively transforming place, activity and meaning. Despite the label, all these significant global megatrends need to be considered so that we are able to interpret the contexts within which mathematics education operates, and to act in ways which allow mathematics education to contribute towards the enlightenment of the individual and the empowerment of the workforce in this era.

Mathematics curricula changes in the first quarter of the 21st Century

Mathematics education systems around the world have reformed or finetuned their respective policies – and thus curricula – over the last 25 years as the 21st Century unfolded (OECD, 2025), as they had in the years before that. A main push factor during this period was the introduction just before the turn of the century (in 1995) of Trends in International Mathematics and Science Study (TIMSS), followed closely in 2000 of the Programme for International Student Assessment (PISA). Both are often regarded as global comparative assessments of mathematics.

Even though TIMSS is administered by the International Association for the Evaluation of Educational Achievement (IEA) in the United States while PISA is a OECD project, the latter should not be regarded as Europe's answer to the American assessment program (even though 27 of the 38 member nations are located in Europe). Planning for the respective first tests began well before we entered the 21st Century, and there were both Western European and North American representation in the UNESCO meeting in the 1980s that launched PISA. Arguably, some features of the PISA design were responses to the way TIMSS was designed. For example, according to Lundgren (2011), one of the objectives of PISA has been to assess students' competencies. Also, TIMSS is known for assessing mathematical knowledge and skills, whereas PISA is widely recognized as a test for students' capability to apply these mathematical knowledge and skills in everyday living (i.e. a test of mathematical literacy). The two assessments also differ in terms of student participants' age groups, as well as frequencies.

Both TIMSS and PISA have been used (often strategically) by governments, education departments, businesses and media in a variety of ways to portray how well or how bad their own mathematics education programs have performed over time and/or relative to others. Not surprisingly, both TIMSS and PISA assessment results have shaped curricula and professional practices. This has especially been the case for PISA, perhaps because PISA assessments take place more often (every three years, instead of every four years for TIMSS). Another possible reason for PISA's influencing

role in curriculum reforms is that assessing 15-year old students' capability to apply the mathematics they have learnt over the primary and secondary school years can be a powerful statistic for relevant stakeholders to capitalise on. Some countries (e.g. Germany and UK) had experienced 'PISA shocks' which paved the way for reforms in their respective mathematics education systems. Rankings of student performance in mathematics and in mathematical literacy, however, have been dominated by East Asian jurisdictions such as China, Hong Kong, Japan, Korea, Singapore and Taiwan over the last 25 years.

Thus, stimulated by TIMSS and/or PISA, and also by traditional pressures from the economic and political spheres, the last 25 years have seen mathematics education curricula in many parts of the world moving towards increased emphases on numeracy (mathematical literacy), data literacy, and 21st Century competencies (including mathematical reasoning) (OECD, 2025). "However, the integration of emerging [competencies], such as empathy and collaboration, remains limited compared to other subjects" (OECD, 2025, p.4).

How might mathematics curricula evolve in the next 25 years or so? OECD (2025) had proposed a set of design principles for mathematics curriculum (re)design, identifying 12 guiding foci, namely, focus, rigor, coherence, transferability, interdisciplinary, choice, authenticity, flexibility, alignment, engagement, student agency, and teacher agency. These foci reflect values which should guide mathematics curricula reforms so that they are forward-thinking, rigorous, and inclusive without causing curriculum overload. While these provide useful perspectives to designing the mathematics curriculum of tomorrow, they are not explicit about what the mathematics curricular themes in the era of digital transformation should / could be. Perhaps there is no need for any change to be made to these themes or topics. Perhaps it does not matter.

More generally, the OECD Learning Compass 2030 (OECD, 2019) proposed a set of themes which are aligned with the 12 foci above, albeit for school subjects in general. Nevertheless, as a subject which all students engage in from primary school years, mathematics provides an ideal channel through which education objectives can be developed in all students. More significantly, the Learning Compass is an expression of what we want students to become, rather than being a guide on curriculum (re-)design. Thus, through an emphasis on student agency, student wellbeing, and competencies, the Learning Compass aims for the next generations of citizens to be ethical, responsible and creative (OECD, 2019).

East Asian mathematics education systems

The six East Asian jurisdictions of China, Hong Kong, Japan, Korea, Singapore and Taiwan were mentioned above for their consistent world-beating performance in TIMSS and PISA. Although their combined population in 2023 is only 19.9% of the world's population (World Bank, 2025), their combined GDP in the same year was 24.4% of the world's, in PPP terms (World Bank, 2025). These six jurisdictions are made up of the world's second and fourth largest economies (China and Japan respectively), and the Four Asian Tigers (Hong Kong, Korea, Singapore, Taiwan). Indeed, East Asia was the most successful economic story in the world post-World War Two. A thriving economy is integral to the survival of each of these East Asian jurisdictions, for different reasons. In this day and age, it is human capital which contributes to such an economy, not agriculture nor manufacture. 40% of the Japanese economy is created in the Greater Tokyo area. Land-scarce Hong Kong and Singapore have both been known for punching above their weights in economic developments; both economies are geared towards financial and service sectors. However, nearly all the six East Asian jurisdictions are experiencing low birthrates and aging populations. The development

of human talent is thus very much emphasized across this East Asian region encompassing the six jurisdictions. They are all amongst the world's top 18 (out of 133) most innovative economies according to the 2024 Global Innovation Index (World Intellectual Property Organization, 2025). In 2025, a quarter of the world's top 20 universities were located in East Asia (two each in China and Singapore, and one in Hong Kong) (Quacquarelli Symonds, 2025).

As the global socio-economic trends evolve and transform in this current era, we can thus expect the mathematics education systems of these East Asian jurisdictions to adjust and finetune accordingly in the interest of developing their respective human talent to their fullest potential. This is not anything novel in the region's mathematics education history. For example, mathematics has since the late 1950s been emphasized in Singapore's school education system to contribute to the country's industrial economy (Kaur, 2019). A more recent example can be seen in the 2011 revision of the Korean mathematics curriculum, in which "mathematical creativity is especially emphasized for national competitiveness" (National Academies of Sciences, Engineering, and Medicine, 2015, p.12).

It is worth noting that 'Western' mathematics has been taught and learnt in East Asian schools for a long time. Hong Kong and Singapore were British colonies, so it is not hard to imagine that schools there had been teaching Western mathematics for more than a century. On the other hand, Western mathematics (and science) was adopted in China during the Qing Dynasty, and later in Japan in the 1850s, for strategic reasons; Western knowledge facilitated an understanding of the psyche of potential aggressors, contributing to self-defence efforts (Siu, 2009).

Of course, even though the six East Asian jurisdictions might be referred to as a group or region, there exist many intra-region differences (see Zhang et al, 2016). That said, these (mathematics) education systems do share one feature in common: all are Confucian Heritage Cultures (CHC) (Biggs, 1996). 2500-year old Confucian teachings in both 'original' and hybrid forms are believed to underlie decisions and actions in these jurisdictions, regardless of how modern or futuristic the society might have become. One might even say that Confucianism is intimately related to education and learning. The very first line of the Analects, when translated into English, reads, "The Master said, "Is it not pleasant to learn with a constant perseverance and application?" (Confucius, 1999, 1.1). Although the Confucian culture is often associated with teacher or student actions such as practice, memorization and high-stakes examinations, we should not forget that these are but surface, visible manifestations of virtues which teachers or students in CHCs practise as part of self-cultivation. Indeed, learning as Confucius saw it is a process of "studying extensively, enquiring carefully, pondering thoroughly, sifting clearly, and practicing earnestly" (Zisi, 1991, 20.19). It is by no means a shallow process, such that "seeing knowledge without thinking is labour lost; thinking without seeking knowledge is perilous" (Confucius, 1999, 2.15). Knowledge and virtues are inseparable (Cheng, 2020), even though this relationship and the practice of self-cultivation are often not referred to explicitly and intentionally.

Given the challenges and opportunities we face in this era of digital transformation, concerned with accelerating advances in digital technologies, with new landscapes in geopolitics characterized by multipolarity, and with continuing threats to the Earth's climate system, how might East Asian mathematics education systems adapt and evolve? Would they continue to have faith in their mathematics education systems and stay the course, or will they respond to emerging megatrends in the world around them and go with the flow? If so, what would that look like?

In this context, I propose four potential institutional responses from the East Asian mathematics education systems to the global megatrends identified. There is no assumption that they will be correct

predictions, not least because world events at the time of writing have arguably never been more volatile since World War Two. Also, like existing and past policies and practices, the extent and nature of any uptake can vary amongst the six East Asian mathematics education jurisdictions. The intention here, then, is to highlight the essence of mathematics education in East Asia in general for the foreseeable future. In this spirit, the four potential future features of mathematics education in this region will be, namely, increased and refined focus on mathematics competencies, more overt emphasis on students' mathematics education well-being, mathematical literacy (numeracy) for the community, and reinforcing the role of teachers as teacher. In the next four sections in which these will be presented, the relevant contextual factors for each will also be discussed. The essence of mathematics education in East Asia underlying these responses will then be proposed, providing the context for explaining whether the mathematics education systems will likely stay the course or go with the flow.

Increased and refined focus on mathematics competencies

It is well understood by now that mathematics learning is not just restricted to learning – or mastering – relevant concepts, knowledge and skills. It is also about developing students' mathematics competencies, in which competencies refers to the “ability to meet complex demands, by drawing on and mobilising psychosocial resources (including skills and attitudes) in a particular context” (OECD, 2005, p.4). After all, the learning of a particular subject should not be just about acquiring the what (through the subject knowledge and skills), but equally importantly, the so-what (that is, what we can do with the knowledge and skills).

At the turn of the century, a group of distinguished mathematics education professors and educators in the United States led by Jeremy Kilpatrick proposed a set of five interwoven and interdependent strands of mathematical proficiency, reflecting

a composite, comprehensive view of successful mathematics learning. Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we think it means for anyone to learn mathematics successfully. (National Research Council, 2001, p.5)

Mathematical proficiency, in the minds of these American scholars, is made up of the five intertwined strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (National Research Council, 2001, p.5).

In the following year in 2002, Denmark released a set of eight mathematical competencies, called the Kompetencer Og Matematiklaering (KOM). They too refer to abilities, that is, the ability to ask and answer questions in and with mathematics, and the ability to deal with mathematical language and tools. These abilities were presented in the form of eight overlapping clusters, made up of mathematical thinking; problem handling; mathematical modelling; mathematical reasoning; representation; symbols and formalisms; communicating in, with, and about mathematics; mathematical aids and tools (Niss, 2013).

When Australia launched a national-level mathematics curriculum for the first time in 2010, a set of four mathematics proficiencies was also introduced. With the four proficiencies named ‘fluency’, ‘problem-solving’, ‘reasoning’ and ‘understanding’ (ACARA, 2009), it is evident how they correspond to four of the five strands of the United States mathematical proficiency. Australian states such as Victoria subsequently adopted the same set of four proficiencies in their respective mathematics curricula.

Turning our attention to East Asia, the current mathematics curriculum in China, ‘Mathematics Curriculum Standards for Compulsory Education 2022’, specifies that developing students’ mathematical competencies is about developing their “ability to observe the real world with mathematical vision”, “to think about the real world with mathematical thinking”, and “to express the real world with mathematical language” (MOE, 2022, p.7). Two features are worth mentioning here. Firstly, the three competencies are broken down into three levels of intermediate goals, at the primary, junior secondary, and senior secondary levels. Secondly, by aiming to “cultivate a scientific and rational attitude” (MOE, 2022, p.6) amongst students, China’s mathematics competencies also pay attention to fostering affective aspects, just like the strand of productive dispositions in United States’ conception of mathematical proficiency. Over the years since then, mathematics teacher professional development workshops and speeches focusing on mathematics competencies have been organised and delivered all over China. The same emphasis could also be seen in Hong Kong and Taiwan.

At the same time, by the end of the first decade of the 21st Century, the Korean curriculum had shifted its focus from content knowledge to core competencies, which included problem-solving, high-order reasoning and thinking skills, and communicating mathematically (Ministry of Education, Science and Technology, 2009).

Over the years, including in the first quarter of the 21st Century, mathematics competencies have not been valued in mathematics teaching as much as it should have been. There is no doubt that mathematics competencies have already been taught in mathematics lessons in different corners of the world, but I would also argue that they are already a part of a teacher’s pedagogical repertoire, often arising from a teacher’s professional concern about the types of competencies that their mathematics students should learn through their mathematics lessons. What has been lacking is more widespread intentional and systematic reference to – and action on – official or standardised sets of mathematics competencies. Perhaps this is because mathematics competencies are often listed in the introductory sections of mathematics curricula only, so that either they get missed out of being incorporated into lesson planning, or teachers interpreted these as not-so-important mathematics education objectives.

However, in this era of digital transformation, our role as educators to foster (mathematics) competencies is all the more important, if not even more so. The advancement of digital technologies has helped many people in many countries to leapfrog, to catch up on access to knowledge. In many ways, knowledge is no longer that prized an asset. What is more crucial is the ability and the imagination of people to apply the knowledge learnt and to create new knowledge to improve quality of life for humankind. Thus, as Frederick Leung commented on Hong Kong mathematics education, “well, even if quadratic equations or the Pythagoras theorem have no application in most of our students’ everyday lives, the process of learning maths is a way to help them to be sharp, critical, and objective. I think this is something that we should really stress” (Ligero, 2023).

While an increased focus on student learning of mathematics competencies is desirable, there is also a need to review the currency of existing mathematics competencies that are embraced by each jurisdiction. After all, the sociopolitical and technological landscapes are evolving very fast. In particular, ongoing directions in AI developments suggest that attributes which make us uniquely human (e.g. empathy, creativity) could soon be exercised by machines. Thus, educational policies and priorities might need to be reviewed more frequently to ensure that mathematics education remains relevant to current and emerging needs of young citizens.

In this regard, the OECD Learning Compass 2030 identified three transformative competencies which it considered to be high-order competencies and which are (currently, at least) uniquely human (OECD, 2019): creating new value through innovating, reconciling tensions and dilemmas, and taking responsibility. It is definitely worthwhile for mathematics education systems to consider how these can be expressed as or through mathematics competencies.

More overt emphasis on students' mathematics education well-being

Mathematics anxiety affects many around the world, including those in high-performing jurisdictions. In fact, of the 15 (out of the 65) participating jurisdictions in PISA 2012 which reported mathematics performance and mathematics anxiety levels that were both greater than the OECD averages (OECD, 2013), all six East Asian mathematics education systems focused upon in this paper are amongst them. Rather than investing in intervention programs which are financially costly and time-intensive with no guaranteed long-term effectiveness, it is more proactive to reduce the likelihood for any student to even experience it. In East Asian cultures where 'saving face' (maintaining *chemyon* in Korea) is a key part of the tradition, mathematics anxiety can go unreported, potentially worsening the situation as affected students might go into depression. This then explains why the idea of fostering mathematical wellbeing (MWB) (Clarkson et al, 2010) had been forward-looking.

Students' well-being also came under the spotlight after the world emerged from the COVID19 pandemic. There had been unprecedented numbers of school closures around the world in 2020 and 2021 at the height of the pandemic, during which time lessons were moved online and students in most countries had very limited opportunities to interact with peers and teachers face-to-face. Concerns about students' well-being were confirmed when schools reopened (UNICEF, 2023).

Longitudinal studies (Richards & Huppert, 2011) had shown that childhood well-being predicts adult well-being. Students' well-being in mathematics learning influences their emotional experiences and their likelihood of continuing with mathematics into senior secondary years, affecting the future STEM workforce which is critical amid global deglobalisation and decoupling trends. Additionally, students' well-being in mathematics learning shapes their engagement with numeracy in daily life, contributing to meaningful citizenship in the era of digital transformation.

At the societal level, governments are learning that both the current political situation in the United States as well as the collapse of the Soviet Union bringing about the end of the Cold War, for examples, have highlighted the importance of vibrant citizenship. Mathematics is one of a few subjects studied by all students throughout the compulsory school years, and students' well-being in their respective mathematics education experiences would thus be a factor worth paying attention to.

As such, in this era of digital transformation in which people might become increasingly detached from group interactions, education systems should focus on fostering and maintaining students' well-being in mathematics learning. Yet, the term 'mathematical wellbeing' (MWB) has been a Western conception, firstly conceptualised by Clarkson et al (2010) in the form of a five-stage developmental model based on Bloom's Taxonomies of Educational Objectives (e.g. Bloom et al, 1956).

Later, educational psychologist Julia Hill conceptualised and validated a different MWB model. Her work was influenced by two main knowledge sources. One regards well-being as a combination of hedonic and eudaimonic traditions, just like Seligman (2011) and Huppert and So (2013). As such, MWB is considered as a measure of how mathematics students were 'feeling good and functioning well' (Hill et al, 2024). The second source of influence came from Valerie Tiberius' value fulfillment theory of well-being. Here, MWB is also a function of the extent to which seven values

are fulfilled, these seven values being *accomplishment, cognitions, engagement, meaning, perseverance, positive emotions, and relationships*. MWB is also regarded as being subjective. That is, well-being reflects “an experience which involves evaluation by individuals of their quality of life according to the standard they set for themselves” (Yang & Zhou, 2017, p.181).

The Hill et al (2024) model of MWB claims to be culturally-responsive; the seven values are considered ultimate values (UVs) each of which is served by any number of instrumental values (IVs) (Rokeach, 1973) allowing for culturally-different constituents. Thus, while the UV *relationships* may be fulfilled in one culture through teacher-student relationships, the same UV can be fulfilled in another culture through, for instance, community relationships that support school (mathematics) learning utilizing close-knitted networks of children within the community.

Notably, there has been no equivalent word for ‘well-being’ in the Chinese, Japanese and Korean languages that are used across the high-performing East Asian jurisdictions. When referring to the ‘Western’ conception of well-being, the Koreans might use the loanword ‘웰빙’ which has the same pronunciation as the English word ‘well-being’. In fact, when asked for the local term for ‘well-being’, Chinese and Japanese scholars would most often offer ‘xing fu gan’ (幸福感) and ‘kofuku’ (幸福) / ‘fukushi’ (福祉) respectively. Koreans also have a more formal term, ‘annyeong’ (안녕). All these scholars would also be quick to add that these terms are not the exact translations, and that they are not identical representations of the Western conception of ‘well-being’.

Perhaps the notion of well-being (in mathematics education) means differently in the East Asian context. For instance, in Korea, well-being might signify for some people there a condition in which one’s mind and body are at ease, one’s abilities are fully utilized across various spheres of life, and one maintains positive social relationships. There is in this interpretation the Confucian doctrines of balance and harmony, as well as of moderation (zhongyong zhidao 中庸之道), when referring to the relationship between mind and body. This interpretation also highlights the continual importance of positive social relationships even in the context of attaining well-being. That is, in the CHC, (mathematics education) well-being is not meant to serve objectives related to personal achievement and gains at the expense of others in the society. If well-being is associated with a good life, then this good life is not necessarily about short-term personal pride, satisfaction or happiness. It thus appears that the nature or source of ‘feeling good’ in the Confucian perspective is less tangible and more intrinsic than we might be more familiar with in the West, with a ‘flavor’ that is more eudaimonic than it is hedonic. Mathematics education well-being in CHCs is thus characterized by eudaimonic (versus hedonic) dominance (Luo et al, 2017). In the meantime, Han and Chen’s (2024) review of the practices of Buddhism, Confucianism and Taoism led to the observation that “eastern views of well-being not only focus on the inner world of individuals but also emphasise the harmony between the individual and the external world” (p.32). Thus, well-being in the CHC is holistically attained as a result of cultivating relevant virtues, and acting in ways which exemplify moderation, harmony and balance. This applies to all aspects of life, including mathematics learning and teaching.

In fact, ‘feeling good’ in the CHC may be understood as referring to a state of mind in the long run. Instead of aiming for the ideal ‘living happily ever after’ in one’s life, the mathematics learner in a CHC would have learnt that happiness and sadness are complementary to each other, that success and failure are complementary as well, for examples. The value and meaningfulness of any one emotion is derived from that of the complementary counterpart, inseparably. Similarly, ‘functioning well’ is a Western conception which is often associated with the individual having a sense of purpose and sense of autonomy in life, whereas the learner in a CHC would be more concerned with

relating well and living harmoniously with the group/community within which they function. Thus, (mathematics education) well-being in the East Asian jurisdictions is not so much representative of the desired state of a student's mind which is attained when certain conditions are met in mathematics lesson planning, delivery and assessment. Rather, it might be considered in the East Asian culture as being an evolving attribute of a person, a product of possibly lifelong cultivation of virtues. For example, there is often the cultural expectation that if one wishes to attain happiness in life, one needs to be ready to strive and work hard. A Chinese idiom talks about 'tasting the sweetness of success only after one has tasted all the bitterness along the way' (苦尽甘来). Another one expresses the message that 'sweetness comes after bitterness' (先苦后甜). A 2024 modern public art installation in Chengdu, China reminded passers-by that if one wants to achieve happiness in life, one has to be prepared to work hard (要幸福，就要奋斗).

Another difference between East Asia and the West in realising well-being lies in how they go about actualising the associated values (i.e. the Confucian virtues in the former, and values such as those in Hill's MWB model in the latter). The MWB model embraces the value fulfillment theory, in which MWB reflects the extent to which seven values are fulfilled. Such a fulfillment can come about from a student's attempts at experiencing, say, *perseverance* or *engagement*, but mathematics lessons do not naturally provide opportunities for these to happen. The teacher has a role in helping students fulfil these MWB values, through lesson planning, classroom interactions and/or assessment approaches. On the other hand, if well-being in mathematics learning in East Asia comes about through daily practising and honing of such Confucian virtue values as *benevolence* and *wisdom*, the student is responsible for this all the time as they live through the mathematics learning process and live through the day. It is what self-cultivation is all about. There is no need for teachers to structure lessons in particular ways, though they can. If the value fulfillment theory leads us to talk about positive and negative well-being, degree of well-being might be considered in the East Asian context, such that a person can boost their degree of well-being through ongoing practising of virtues. Furthermore, unlike the seven MWB values which might not apply to all school subjects, the Confucian virtues constitute a part of any student's life so anyone who grows up in an East Asian environment already knows what each virtue entails. The mathematics learning process and other daily-life events simply provide them with opportunities to further cultivate these virtues.

Indeed, these two distinctions of well-being in mathematics education in East Asia and in the West are reflected in an ongoing longitudinal research study in Chengdu, China, documenting the roles of virtue values in the development of what it calls 'mathematics education well-being' (MEW) (The Third Wave Lab, 2025) across grade levels. The subtle but intentional adoption of 'well-being' (c.f. 'wellbeing' in the MWB model) is meant to emphasise how wellness is defined by the moment-by-moment quality of being, that well-being itself is dynamic and evolving with life experiences.

The MEW model also shares a feature present in the earlier Clarkson et al (2010) conception of MWB: the notion of (mathematics education) well-being as existing along a continuum marked by stages of development. This is in line with the East Asian belief in degrees of well-being. A student's well-being is progressively developed and its intensity increased over time, as the student continues to practise relevant virtues across a range of life and mathematics learning experiences. The development of MEW is regarded as beginning with the noticing stage, then progressing through the welcoming, valuing, integrating, and then, flourishing stages, as shown in Table 1.

Table 1. Stages of development of Mathematics Education Well-being (MEW)

Flourishing Finding satisfaction, joy and even playfulness with mathematics Sharing my joy with others, inspiring others, motivating others, helping others Learning, being, and doing have become one Understanding that mathematics is a symbolic system of the human civilisation
Integrating Confidence in my level of skill and competence Steady and focussed when faced with setbacks, showing inner clarity Identifying mathematics that is embedded in other school subjects Relating mathematics to my life objectives, identity, or future plans
Valuing Appreciating and enjoying mathematical activities Exercising self-compassion Seeking out meaningful learning goals beyond grades, such as mathematics competencies Being aware that mathematics has developed in different cultures Treating mathematics as an analysis tool or an aesthetic object Achieving high levels of mathematical competence
Welcoming Involving in mathematical activities invokes a positive response Experiencing pleasure in pursuing mathematical activities Treating mathematics as unified, coherent scientific body of knowledge
Noticing Passively accept mathematics learning contexts, disinclined in seeking them out Regarding mathematics as a collection of mathematical activities Accepting that mathematics is a subject that I need to study at school

Student well-being in mathematics learning in the East Asian context might be operationalised differently from how it is in the West, but more active and visible fostering and developing of this form of well-being in mathematics education will gain in significance, as different jurisdictions seek to optimise human talent especially for its STEM-based industries. There is also likely an expectation from the population for students' well-being to be taken care of by school authorities, as the current generalisations of young parents grew up in a global period of relative prosperity and peace, and appear to be more embracing of lifestyles that emphasise quality of living. Also, as increasingly more parents (and even teachers) in East Asia are able to distinguish well-being from 'purely having fun and being happy', there will be a greater public acknowledgement of the value in investing in promoting MEW amongst students.

Mathematical literacy (numeracy) for the community

The ongoing global demand for STEM-qualified workers is expected to continue in the foreseeable future. Both the COVID19 pandemic and the decoupling of the Chinese and United States economies have highlighted the importance of national self-sufficiency of manufacturing plants and supply chains. Thus, a range of STEM-qualified workers are needed across industries and sectors. This can potentially affect East Asian jurisdictions negatively, except China, given constraints in either population size or land size, or both. While countries like Australia, Canada and the USA possess the soft power to attract qualified migrants to address supply shortages in their workforce, nearly all East Asian nations have either remained relatively homogenous in terms of ethnicity (so far) and/or they are not as attractive a working destination. The expectations will be on their own school education systems to nurture sufficient STEM workers. China, for example, is actively encouraging more students to select vocational courses at the end of the junior high school education, introducing 40 new vocational majors at the end of 2024 (Ng, 2024) which cover critical subjects such as digital technologies and advanced (including aerospace equipment) manufacturing. Moves such as this is crucial as the Chinese job market cannot absorb quickly enough the ever-increasing numbers of university graduates annually, which is projected to exceed 12.2 million in 2025 (Reuters, 2024). Singapore has also been implementing similar policies, such that vocational students have been constituting “over 60 percent of the cohort who go on to postsecondary education” (National Center on Education and the Economy, 2023).

At the same time, research evidence is emerging (e.g. Menichelli & Braccini, 2020) that the accelerating developments of digital technologies especially in AI and in social media (threaten to) weaken humankind’s critical reasoning capabilities. Being numerate is an important component of such a capability, providing human consciousness with the tool to process and interpret numerical and spatial data, based on which creative and wise decisions are made in the face of ever-prolific mis- and disinformation. Thus, while not everyone in the population needs to be a mathematician, we certainly need everyone to be numerate sufficiently to be able to navigate through the demands of daily living and of workplace tasks and responsibilities.

However, current workforce numeracy levels across the East Asian jurisdictions have been mixed. Of the three participating East Asian countries in the latest OECD Survey of Adult Skills 2023, Japan led the world with the lowest proportion of low performing adults (10%), who were not able to “access, act on and use mathematical information, and evaluate simple claims, in tasks set in a variety of authentic contexts” (OECD, 2024a, p. 60) for example. But Singapore and Korea’s share of low performing adults were both higher than the OECD average. Of course, this does not reflect the quality of mathematics and numeracy education in East Asia (which is shown through performance in PISA instead), but it does reveal the impact of many of the East Asian school education systems having been set up only after the late 1950s on the numeracy levels of their current workforce. For example, it would have taken a while for school attrition rates to be under control. Japan was probably the odd one out, since education reform (with substantial support from the United States) was amongst the most urgent task right after 1945, so that the Japanese economy could rebuild itself and thrive quickly.

There is certainly a need for relevant East Asian jurisdictions to boost adult numeracy levels, especially the older ones in their 50s and 60s, given that retirement ages might also be extended in these economies. This may take the form of government-supported up-skilling or re-training courses (e.g. Singapore’s SkillsFuture program for citizens aged 40 and above) with a focus on numeracy. However, these courses might not reach those very citizens who are already low-performing in numeracy (and probably literacy too), so community-level courses can be organised to support life-

long learning motivations of older citizens. Japan, one of the most aged-populated countries in the world, possesses rich experiences in this aspect (see Le & Billett, 2022). As Japan is also an East Asian country, it will have much valuable culturally-informed insights and knowledge to share.

So, what is numeracy? Also known as mathematical literacy (Gal et al, 2009) in some contexts, OECD's Programme for the International Assessment of Adult Competencies (PIAAC) defined it as "the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of a range of situations in adult life" (Gal et al, 2009, p.14). Other definitions (e.g. Tout, 2020; Willis, 1990) are similar, thus emphasizing the role of mathematics in the development of the numeracy competency.

Unfortunately, numeracy has been regarded as second-class mathematics in some societies, instead of being recognised more highly in a way it deserves (Goos & O'Sullivan, 2023). It has even been called 'veggie maths' in some countries. Many people have mistaken numeracy as arithmetic, as a basic and 'easy' part of mathematics. A reason for many societies' underestimation of what numeracy involves is the belief that people are generally numerate (knowing how to count) even though they may not be mathematicians (dealing with calculus and geometry). It does not help either that PISA's focus on numeracy or mathematical literacy is not widely understood in the society; many a person still regard it as another TIMSS or the European version of the American TIMSS.

In fact, as can be seen in Figure 1 below, numeracy is concerned with not just mathematical knowledge and contexts, but dispositions and tools as well. In particular, a numerate person possesses positive dispositions towards the application of mathematical knowledge in real-life, authentic contexts, drawing on representational, physical and digital tools.

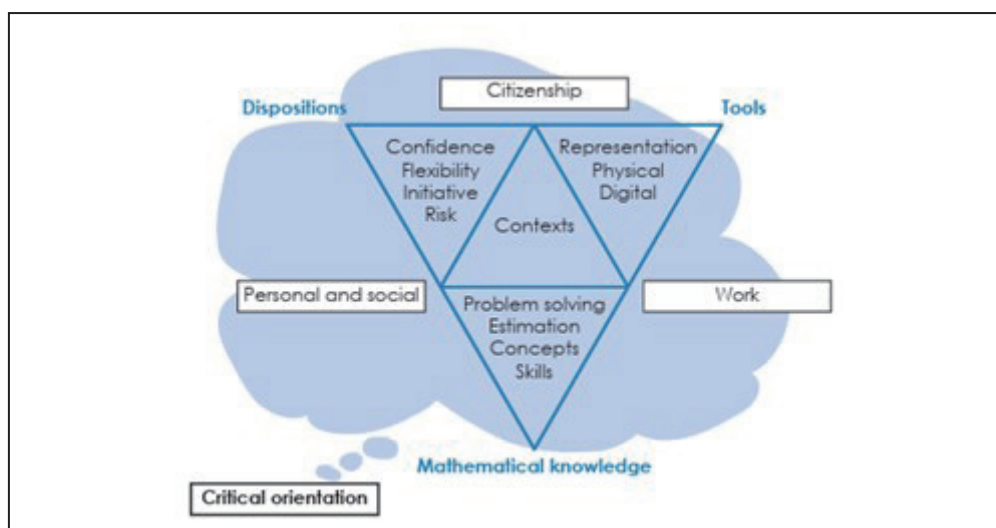


Figure 1. Goos et al's (2012) model for numeracy in the 21st Century

As such, we can understand why there are countries such as Australia which occupy a relatively high ranking in TIMSS exercises, yet perform dismally in PISA. Possession of a strong mathematical knowledge is not a guarantee for any student that they can apply it in novel situations or in daily life. Frederick Leung, the then President of the International Commission on Mathematical Instruction, emphasised in an interview in 2023 that students need their teachers to build the bridge between the theoretical aspects of mathematics and the array of contexts in which they apply: "we teach the theoretical side of maths and then some trivial application without relating it to the real

world” (Ligero, 2023). This remark highlights how numeracy is not just applications of mathematics, involving as it does non-trivial, authentic and meaningful applications.

Numeracy could be taught through an insertion of authentic applications in school mathematics. The ‘Realistic Mathematics Education’ (RME) movement which began in the 1960s in the Netherlands is commonly regarded as the pioneer reform program incorporating this approach (van den Heuvel-Panhuizen, & van Zanten, 2020). South Africa’s approach to embracing numeracy is through the introduction in 2006 of a standalone subject called ‘Mathematical Literacy’ in Grades 10-12, as an alternative to mathematics. While this was in response to a political concern that all secondary school leavers should learn some kind of mathematics, it also probably contributed to public opinion that numeracy is an ‘easier kind of mathematics’ (Goos & O’Sullivan, 2023).

More recently, mathematics education systems in Australia, Canada and Ireland have adopted a ‘numeracy across the curriculum’ approach. The Australian Curriculum (ACARA, 2025) names numeracy as one of the seven General Capabilities, to be “developed within the context of the learning areas [that is, school subjects]” (ACARA, 2025). In Ontario, Canada, mathematical literacy is part of cross-curricular and integrated learning, in which relevant “perspectives, themes, and skills are intentionally incorporated by educators, on an ongoing basis, into teaching and learning across all subjects and disciplines of the curriculum” (Ontario Ministry of Education, 2025). Ireland is also doing something similar, incorporating the N framework (O’Sullivan et al, 2019) in which numeracy knowledge is seen as bridging subject specific knowledge and pedagogical content knowledge.

How might East Asian mathematics education systems respond to this distinction between mathematics and numeracy / mathematical literacy? Amongst the six East Asian jurisdictions, Japan and Korea are members of OECD that hosts the three-yearly cycles of PISA, which as we know assesses students’ mathematical literacy. As such, both these countries are probably more cognizant of the concept, and indeed, have developed local terminologies with which to refer to mathematical literacy. The Japanese language for mathematical literacy is 数学的リテラシー. In fact, the Korean language differentiates numeracy (수리력) from mathematical literacy (수학문해력), with the latter alluding to making sense of the language (of mathematics). In Japan,

aspects of mathematical literacy emphasising real-life contexts, positive dispositions, and using mathematical knowledge for problem solving in complex social settings are being infused into the regular ... school mathematics curriculum. (Goos & O’Sullivan, 2023, p.354)

The Japanese approach is thus in contrast to the one adopted in Australia and Ireland, which incorporates other school subjects. It is also in contrast to South Africa, in which numeracy is taught as a standalone subject. In fact, the Japanese approach can be seen as an extension (or the highlighting of one) of what is already being taught through mathematics. This was first introduced in the Japanese mathematics Course of Study (i.e. curriculum) as early as in 1958, and updated versions have since been featured in successive curricula. The current primary curriculum (Isoda, 2010), for example, states that the objectives of teaching mathematics are

to help pupils acquire basic and fundamental knowledge and skills regarding numbers, quantities and geometrical figures, to foster their ability to think and express with good perspectives and logically on matters of everyday life, to help pupils find pleasure in mathematical activities and appreciate the value of mathematical approaches, and to foster an attitude to willingly make use of mathematics in their daily lives as well as in their learning. (p.18)

Numeracy or mathematical literacy is often not discussed in the other four East Asian education systems, however. Yet, East Asian jurisdictions have been leading other countries in numeracy performance measured by the cycles of PISA exercises. Given that PISA emphasizes numeracy where-

as TIMSS emphasizes more on mathematical knowledge and skills, and yet these jurisdictions came up top nevertheless, they have demonstrated that one can be as proficient in numeracy as they are in mathematics. That Korea and Singapore have sizeable proportions of their workforce not being numerate enough (see discussion earlier in this section) is not a reflection of the quality of their current mathematics education, but a legacy of historical developments, as discussed above.

The paradox then is, how do the East Asian mathematics education systems rank as world-best consistently in PISA assessments in the last 25 years if the very subject it purports to assess seems to be non-existent in East Asian mathematics curricula, or (in the case of Japan) an add-on to the curriculum minimally? One way of interpreting this might be that students' ability to apply their mathematical concepts and skills in novel, day-to-day contexts has been developed and honed through mathematics lessons in school, similar to the Japanese approach outlined above. For example, in the Singapore secondary mathematics syllabus (Singapore CPDD, 2023),

students are expected to be familiar with the following contexts and solve problems based on these contexts over the four years of their secondary education:

- In everyday life
- In personal and household finance
- In interpreting and analysing data from tables and graphs. (pp.13-14)

One might argue that mathematics lessons everywhere have as one of their objectives students' application of mathematics. The same can be said of mathematics curricula in the East Asian jurisdictions. The difference, however, is that there has always been a cultural expectation that applications is an objective for all kinds of learning. There is a Chinese idiom which translates to 'learn for the purpose of applying' (学以致用). Zhu Xi, a Song Dynasty Confucian scholar, taught that the objective of learning is in its application (Tang & Shen, 2022). Also, Hong Kong mathematics educator, Frederick Leung, emphasized that teachers have the responsibility to "point out the relationship between a real-life problem and the math that they learn in school" (Ligero, 2023).

In other words, even without the need for a specific term to represent the concept of numeracy, East Asian mathematics education has been about teaching for both theoretical knowledge and applicational capability. That they happen to excel in both, as seen in their performance in TIMSS and PISA, might be regarded as a bonus. It is likely that this approach will continue.

Reinforcing the role of teachers as teacher

Given the development of artificial technology in recent times and in the foreseeable future, it can be tempting for some people to feel that teachers can be replaced by the seemingly always-patient, always-knowledgeable and always-ready AI bot. One of the most impactful breakthroughs towards the end of the first quarter of this century would have been the 'ability' for AI applications to understand natural language, so that user inputs and queries need not be in coded, computer language like it used to be. AI applications in the mathematics classroom is not just in the delivery of mathematics lessons; there have been studies as well on how we can develop AI's capability to plan mathematics lessons (e.g. Baytak, 2024; Karaman & Goksu, 2024). However, AI-generated lesson plans are not (yet?) able to customise teacher practices to the unique characteristics and needs of the diverse profiles of students in any mathematics class (Gurl et al, 2025). The AI lesson plan remains at best to be a draft lesson plan for the human teacher to customise to their own mathematics learners.

While AI applications have become very good at not just providing knowledge but also explaining it, it is not yet able to probe our understanding in order to ask us (follow-up) questions as an assessment strategy. So, even if teachers are to outsource some of their explanation work to AI applica-

tions, they will need to be even better at the art of asking questions that constitute formative assessment and which promote learning.

After all, most of the promised or projected benefits of AI have not yet been shown to be true, including personalisation of learning experience, real-time feedback, adapting to individual students' needs, and creation of interactive learning environments.

The discussion above provides a few examples of limitations of AI and other digital technologies which render the human (mathematics) teacher indispensable. Even if AI is to be able to take over some of the traditional roles and tasks performed by teachers, it is not necessarily a bad thing from the efficiency perspective; it frees up more time and opportunities for the human teacher to focus on higher-order pedagogical tasks. These higher-order tasks include being able to critique the content of AI-generated lesson plans to finetune and customise the lesson exposition approach, the choice of specific questions for classroom examples, strategies to engage students who are 'faster' or 'slower' than the majority of the students in class, and how the learning can be optimised for students who are affected by some personal experiences on the day of the lesson. The human teacher is also required to know what different questions to pose to different students during the lessons, capitalising on their respective zones of proximal development (Vygotsky, 1978). These are all teaching skills which require professional experience, judgement and decision-making. In the era of digital transformation, then, the teacher does not become an assistant to AI. Rather, with AI taking over the responsibility of basic pedagogical tasks, the teacher's role as teacher is even more magnified.

Current world events and phenomena have created other pressures on the development and evolution of mathematics education, of course. However, they are not likely to impact on the East Asian education systems as much. For example, the shortage of qualified mathematics teachers as experienced in many countries in the West is a phenomenon which the six East Asian jurisdictions have not needed to confront. This had led to a sizeable proportion in the secondary teaching workforce of out-of-field mathematics teachers, especially in countries like the United States and Australia (Hobbs & Porsch, 2021). East Asia has avoided this due partly to the relative attractiveness of tertiary mathematics courses, and also due partly to the high social status accorded to teachers. In fact, only the top students in the pre-tertiary high stakes examinations in Korea can apply for pre-service teaching degree courses, further elevating the status of the teaching profession in the country. Another example would be the relatively large achievement gaps in numeracy experienced by Western mathematics education systems. Thus, according to PISA 2022 results, the East Asian mathematics education systems considered in this paper constitute five of only seven jurisdictions having 20% or less of their students performing below Level 2 ("the baseline level of proficiency students need to participate fully in society" (OECD, 2023, p. 89)). The sixth East Asian jurisdiction (China) did not take part in PISA 2022. Nevertheless, East Asian policy planners and thought leaders would no doubt be watching these developments to minimize or prevent similar issues from becoming problems in the respective mathematics education systems.

These four themes discussed above (increased and refined focus on mathematics competencies; more overt emphasis on students' mathematics education well-being; mathematical literacy (numeracy) for the community; reinforcing the role of teachers as teacher) represent what I believe to be the dominant features of mathematics education in East Asia over the next few years as we embark on the second quarter of the 21st Century. Specifically, the increased importance of emphasizing and teaching mathematics competencies was identified, together with an expectation of East Asian mathematics education systems to review and update their respective lists of mathematics competencies to ensure that they remain relevant and empowering for mathematics students. At the same time, the fostering and maintenance of mathematics education well-being amongst students repre-

sent a proactive approach to preventing mathematics anxiety while also enabling students to thrive in their mathematics learning experiences. Mathematics education well-being in the East Asian context is expected to continue to display eudaimonic dominance, however. Teachers can/will be expected to provide students with opportunities to practise and cultivate virtue values, opportunities which manifest in forms more familiar to typical mathematics classroom scenes. As such, it is meaningful in the East Asian context to describe mathematics education well-being in the form of developmental stages. Thirdly, while current mathematics education programs in East Asia might nurture students proficient in both mathematics and numeracy, many of the jurisdictions might need to boost the numeracy levels of their current adult workforce. Lastly, an argument was also made that when many believe that AI might take over teachers' jobs in this era of digital transformation, it is paradoxically even more crucial for teachers to engage in what they are trained to do best, that is, to teach.

Confucian Heritage Culture in East Asia

Some or all of the four potential evolving features of East Asian mathematics education as a response to emerging global megatrends may or may not resemble how the other systems around the world responded. Of interest in this paper is how they reflect the essence of mathematics education in East Asia in the era of digital transformation.

In mathematics education research, Leung (2006) had proposed that the prevailing culture in East Asia might be key to understanding the way mathematics is taught and learnt in the region. In particular,

one possible root ... is the underlying Confucian cultural values common to the East Asian countries. It is well known that in the Confucian culture, there is strong emphasis on the importance of education. Under the influence of this philosophy, East Asians consider learning or studying a serious endeavour, and students are expected to put in hard work and perseverance in their study This high expectation on the student to achieve provides an important source of motivation both for teachers to teach conscientiously and for students to learn well. (p.43)

More recently, in 2023, Leung was quoted in an interview:

In the Confucian culture there is a lot of emphasis on effort. So, if you are not good in mathematics, you put in more effort In fact, if we looked at the international comprehensive [sic] studies in maths education, like PISA carried out by the OECD, or the IEA TIMSS projects, there is a very robust finding, and that is, countries under the Confucian heritage culture perform consistently well in mathematics. So, I think this is evidence that the culture actually has a lot of influence on students' mathematics teaching and learning. (Ligero, 2023)

Before we examine Confucian values in East Asia in the next few paragraphs, let us first turn our attention to a probable response to the quote above. We may observe that most if not all other (mathematics) education systems also value *education* and *effort / hardwork*, albeit to different degrees. This is probably the reason why Carol Dweck's (2007) growth mindset theory and Jo Boaler's (2015) mathematical mindset theory might have been embraced in these other mathematics education systems. Unfortunately, in many other countries these societal and research-informed academic values may not translate into nor inform pedagogical values of the day, especially when these are not aligned with the local culture. Another issue facing many (mathematics) education systems outside East Asia has been the adoption by schools of "faddish but unproven maths teaching methods" (Hunter et al, 2024, p.3). So, in many ways, it helps that across East Asia there is generally a harmony between societal and pedagogical values, specifically values inspired by Confucian virtues.

Confucian virtue values

Nevertheless, it does beg the question of the extent to which values associated with Confucianism, such as *practice* and *achievement*, are unique. Leu's (2005) research with Taiwanese teachers of mathematics highlighted the valuing of Confucian attributes and values such as *success*, *scores*, *examinations*, and *obedience to authority*. Leung (2006) identified values such as *education*, *effort*, *practice* and *examinations*. Seah (2025) considered the virtue *filial piety* as being the terminal – or ultimate – value for instrumental ones such as *effort* and *success*.

This notion of terminal values deepens our understanding of how the Confucian virtues underlie daily mathematics pedagogical practices. It helps to explain why other cultures can try – or are already – embracing values such as *achievement* or *practice* in their respective mathematics classrooms but do not experience the same outcomes as East Asian mathematics lessons. So, a student in East Asia might value *practice*, spending much after-school time working on mathematics problems and/or sitting for time-trial mock assessments, either because they believe that it would develop their virtue of *wisdom*, or because they regard it as their responsibility as a parent's child or a teacher's student to do well, fulfilling the virtue of *filial piety*. Yet, another student in a non-CHC culture might value *practice* and engage in similar tasks as their East Asian peer, but doing so with the intention to satisfy school criteria for promotion to the next school grade and fulfilling the student's valuing of *accomplishment*, for example. Thus *practice* in these two scenarios are serving different terminal values, *wisdom* or *filial piety* in the former, and *accomplishment* (not a Confucian virtue) in the latter. The difference in motivational strength between these two sets of terminal values can be appreciable. There are at least two ways of accounting for this difference. Firstly, the Confucian virtue values are 'bottomless', so one might keep fulfilling them as part of self-cultivation without expecting or looking for an end-point. On the other hand, one can specify a goal for the valuing of *accomplishment* (e.g. promotion to next grade level, completing all the questions in the textbook) which means that tasks fulfilling this valuing can have a clearly-defined end-point. Secondly, most if not all the Confucian virtues are obligatory in nature, in that students in CHCs are educated from young that they have role responsibilities and obligations. This can have an impact on the strength of student motivation, given that "the meaning system of Confucian heritage culture is crucial for the understanding of learners' motivation among East Asian societies" (Chen, 2023, p.4).

The Confucian virtue values are also aligned with many mathematics teaching approaches. For example, active learning and understanding are often encouraged amongst our students. In this context, Confucius was quoted as saying that when they are taught a perspective, a student should be able to actively understand other related perspectives (Confucius, 1999, 7.8). As another example, group discussions can be a productive pedagogical approach in East Asian mathematics classrooms. "Mathematics knowledge is co-constructed by the group or the class. Having subgroups of three/four students (instead of sitting in pairs) decreases students' hesitation in speaking up and increases their participation in class" (Sum & Kwon, 2020, p.10) as they work towards consensus for their group responses.

Sum and Kwon's (2020) research in an elementary school in Seoul over 6 weeks provided another example. In this instance, the teaching strategy of revoicing was the focus:

unlike the traditional discourse structure of IRF/IRE, teacher revoicing repeats and rephrases students' utterance without giving directly response/evaluation, which may cause embarrassment and ethnical emotional strain to students. Students are therefore not afraid to make mistakes for the fear of losing *cheymon* or being ashamed about failing to meet the social expectations of the teacher and their peers in the class. (Sum & Kwon, 2020, p.9)

The strength of traditional, societal values expressed through the Confucius virtues should not be underestimated. The potential workability of community-level numeracy programs is derived from East Asians' valuing of *lifelong learning* serving the terminal virtue value of *wisdom* (see Cheng, 2020). As another example, the 'double reduction' policy introduced in China in 2021 has not led to the demise of after-school tuition industry. Parents (and even teachers) (Li, 2024) who valued *achievement* and the Confucian virtue *wisdom*, including those whose children genuinely benefit from after-school, disciplined lessons, continued to support the many education companies. In 2024, "the New Oriental Education & Technology [Group's] ... schools and learning centres increased by 277 year-on-year, while its revenue from new education businesses increased by 50.3% year-on-year. ... TAL Education Group, Xueda Education Group, and Only Education Group also reported performance growth of 35% to 50%" (Li, 2024). The same may be said of Singapore, which has introduced several policies in recent years to de-emphasise the role of high-stakes examinations, so that time is freed up for students to pursue extra-curricular interests, and also to recognise non-academic forms of achievement and talent in scholarship selections and university entries. Yet, parents could be seen enrolling their children for more tuition classes with the extra time made available, as a result of which expenditure on private tuition reached a record SGD1.8 billion (USD 1.4 billion) in 2023 (Tushara, 2025). The situation in Korea is no different: 80% of students across all levels of formal schooling attended after-school tuition classes, recording a total expenditure of 29.2 trillion won (USD 20.6 billion) (Statistics Korea, 2025).

The powerful cultural influences exerted by these Confucian virtue values can also be observed in the approaches adopted by East Asian mathematics education systems as they respond to global megatrends in the next few years. For instance, we can expect that curricular focus on mathematics competencies will be complemented with active mathematics teacher professional development programs to ensure that classroom teachers are equipped to effectively deliver the curriculum. This reflects expectations of leaders in Confucian societies to assume the responsibility to look after the welfare of those they take charge of (Ma & Marquis, 2022). This exercising of responsibility is a part of what it means for practicing *benevolence* in one's life. As another example, the importance which the East Asian nations attach to international comparative assessment exercises such as TIMSS and PISA, and the way their achievements are celebrated, both reflect the valuing of educational *success* across these nations. A third example can be seen in the way CHCs remain cognizant of the teacher's role – which is to teach – in the face of technological applications to help teachers plan and deliver lessons as well as to assess learning, a reflection of Confucian valuing of respecting social hierarchal structures and authority.

The influence of Confucian beliefs and practices is arguably most stark when considering East Asian mathematics education systems' response to a trend in the West of fostering students' well-being in mathematics learning. In fact, not being able to identify a word in the Chinese, Japanese and Korean languages (the main languages in the East Asian region) to represent 'well-being' suggests that students' well-being has been developed in East Asia quite differently. In East Asia, developing a student's mathematics education well-being involves students' self-cultivation, dedication to long-term gains (instead of quick-fix solutions), and maintaining harmony between self and the environment.

It is thus very likely that Confucian virtue values will continue to influence and guide mathematics education policies and practices in the second quarter of the 21st Century, as they had over the last 25 centuries or so. Yet, there are also pragmatic elements in the ways in which the East Asian systems respond to the demands on mathematics education in the era of digital transformation. For example, despite the several authoritative models of mathematics competencies (e.g. National Re-

search Council, 2001; Niss, 2013), East Asian nations have identified competencies which are relevant or meaningful to local contexts and values. For example, both the Chinese and Korean mathematics curricula emphasise students' ability to express themselves through mathematical language, thereby communicating mathematically effectively.

Another pragmatic approach is in the ways in which students' well-being in mathematics learning is being promoted amongst the East Asian jurisdictions, when it might be argued that well-being would have been attained naturally to different degrees as one engages in self-cultivation of virtues in daily and school lives, requiring no additional attention. Yet, perhaps in the context of high teenager depression and suicide rates, and/or perhaps with evolving expectations of millennial parents, there are signs of mathematics education well-being promotions in schools allowing for a certain degree of realising students' personal accomplishment (see Liang et al, 2022; Zhong et al, 2024), even though these are at all times still means to fulfilling loyalty and filial piety obligations.

The essence of East Asian mathematics education in the current era of digital transformation may thus be a pragmatic form of Confucianism. After all, Confucian principles and values are flexible enough to accommodate 'variants' (e.g. Ma & Marquis, 2022). Confucianism's "ultimate goal of personal and social life [which] is to pursue harmony" (Ma & Marquis, 2022, p.21) is evident through harmony between mathematics concepts / skills and competencies, deriving well-being in mathematics learning through living harmoniously with the external environment, harmony between mathematics and numeracy, as well as harmony between being a teacher and being user of digital technologies.

This emphasis on harmony and balance might well serve mathematics students in East Asia well. An analysis of the PISA2018 results had found that "students would be more likely to get both high mathematics achievement and high SWB [subjective wellbeing] in a society with high long-term orientation and low indulgence culture" (Meng & Liu, 2022, p.6), and according to Hofstede et al (2010), these two societal features are to be found in East Asia:

different from the happy culture emphasizing on the high emotional arousal, Confucian heritage culture seeks for the balance state between extreme positivity and negativity, such as serenity, inner harmony, and mindfulness The emphasis on balance may contribute to a better balance between high mathematics and high SWB. (Meng & Liu, 2022, p. 7)

To the extent that the predicted responses offered in this paper are taken up by the East Asian mathematics education systems, this paper suggests how culture continues to exert a strong influence on how (mathematics) policies, curricula and classroom delivery might be changed in response to evolving phenomena around the world. The notion of terminal (versus instrumental) values helps us to understand why mere adoptions of practices in cross-cultural scenarios might not work in one's own cultures. That Confucian virtue values acting as terminal values are 'bottomless' might explain observed incidences of motivation strength, and other cultures should be able to identify these from amongst the traditions and/or religions which are prevailing in their respective societies.

It is important to remind readers that it is not the intent of this paper to promote pragmatic Confucianism-informed mathematics pedagogies outside East Asia as a means of improving mathematics performance. After all, one of the ultimate values embraced by Confucianism is *loyalty*, requiring that self-cultivation of virtues facilitates service to the nation state (xiushen qijia zhiguoping tianxia修身齐家治国平天下) rather than for personal glory or gains. This idea might be in conflict with contemporary beliefs and traditions in many other countries elsewhere, beyond Hofstede's (2011) conception of the *collectivism – individualism* value continuum. It would be a form of cultural colonization if the Confucian approach is promoted, which is undesirable, not to mention impossible.

Rather, through the contents shared in this paper, how East Asian mathematics education jurisdictions might respond to global megatrends as the world enters the second quarter of the 21st Century, is explained through the perspective of pragmatic Confucianism, which continues to underlie culturally-based reasoning and problem-solving regardless of how much the societies might have ‘moved on’.

Conclusion

The second quarter of the 21st Century will be an uncertain one; two of the three identified global phenomena impacting on all of us – accelerating advances in digital technologies, and emergence of a multipolar world – are only beginning to unfold right before us at the time of writing this paper. That the six East Asian mathematics education systems in China, Hong Kong, Japan, Korea, Singapore and Taiwan have been dominating both the TIMSS and PISA rankings in the first quarter of this century – and often improving on their own performance in the previous cycles – suggest that they must have been doing something right in both mathematics and mathematical literacy (or numeracy) education. It is not without some unintended side-effects. Nevertheless, making these mathematics education systems even stronger in the next quarter of the century is even more urgent, given the predicted global demand for human talent which for these East Asian jurisdictions would probably be dependent on their local populations. This period (and probably the one beyond that) should witness Asia as being a global economic powerhouse, and local mathematics education systems should complement and support governmental policies aimed at achieving or managing this phenomenon. In this context, pressures on the East Asian mathematics education systems have been projected to be related to mathematics competencies, mathematics education well-being, numeracy (i.e. mathematical literacy), and the role of the mathematics teacher. We can expect to see a greater emphasis on mathematics competencies in the years ahead, with the content continually evaluated for currency. We can also expect a more explicit emphasis of promoting and developing students’ mathematics education well-being, with an approach that acknowledges the oft-invisible self-cultivation of (Confucian) virtue values as well-being builds up over stages and time. This period will also demand each of the six jurisdictions (and elsewhere around the world) to maintain a functionally numerate population. While these mathematics education systems have been facilitating numeracy learning in mathematics lessons well, innovative and effective programs will still need to be generated to promote numeracy learning amongst school leavers, working adults and aged citizens. This region will also likely reinforce the role of teachers to teach explicitly, just when public perceptions might tilt to other views given the (promised) capabilities of digital technologies, including generative and agentic AI.

These mathematics education practices forecasted for the East Asian jurisdictions reflect how perspectives and worldviews in the local societies continue to be guided by pragmatic Confucianism, as they probably have been for the past 2,500 years. Thus, if the essence of mathematics education in East Asia is pragmatic Confucianism, will this stay the course or go with the flow in this era of digital transformation? As might be discerned in this paper, the answer is probably a bit of both. Confucianism will remain relevant, but we can also expect it to evolve and adapt pragmatically to new contexts and environment within which mathematics education systems continue to ensure that students leave school and contribute to the society feeling knowledgeable, skillful, confident and motivated.

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Plenary Lecture 2

**ADDRESSING ALIGNMENT OF COMPETENCY-BASED MATHEMATICS
CURRICULUM: CHALLENGES AND OPPORTUNITIES**

Yoshinori Shimizu



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



ADDRESSING ALIGNMENT OF COMPETENCY-BASED MATHEMATICS CURRICULUM: CHALLENGES AND OPPORTUNITIES

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Curriculum reforms in school mathematics have taken place around the world within contexts and under educational policies that vary significantly. In this plenary lecture, I present a series of studies that demonstrate how intended curriculum, implemented curriculum, and attained curriculum, function together in school mathematics during a transitional period of the curriculum reform. While the pathways for disseminating the new goals can be multi-layered and complicated, the analyses of textbooks aligned with the revised curriculum standards revealed that the way in which the new goals are reflected in textbooks varies and that the implementation of the intended curriculum with the revised textbooks is largely left to the capacities of teachers. The case studies of classroom practice, at the level of implemented curriculum, suggest that teachers' interpretation of intended curriculum, or perceived curriculum, does matter to student learning in the classroom and hence to the attained curriculum. Throughout the lecture, it is emphasized that teachers are key players in achieving the alignment among the three levels of curriculum.

Keywords: school mathematics, curriculum, alignment, competency, textbook, ICMI Study 24

Introduction

The curriculum is a core component of school education that needs to be examined critically and periodically to achieve proper goals of education in a changing society. School mathematics curriculum reforms have taken place in many countries around the world within contexts and educational policies that vary significantly. Both mathematics teachers and mathematics educators have been involved in curriculum reform in various ways. As Kilpatrick (2023) points out, however, “*Despite enormous amount of curriculum development, we do not have an enormous amount of curriculum development research.*” (p.29). The International Commission on Mathematical Instruction (ICMI) launched ICMI Study 24 that aimed to explore the current status of research on school mathematics curriculum reforms and to identify the issues related to it. ICMI Study 24 offered a unique opportunity and a platform to examine past and present mathematics curriculum reforms in different parts of the world, from a macro perspective and meta level and to investigate larger questions of who or what sectors of society drive and most influence curriculum reforms, what reforms precisely are taking place, how are these being implemented, and if they are deemed successful (or not), what count as success (Shimizu & Vithal, 2023a).

One of the major lessons learned from ICMI Study 24 was that school mathematics curriculum reforms can be regarded as a wide-spread practice but under-researched topic in mathematics education (Shimizu & Vithal, 2023b). There are both driving forces and barriers affecting curriculum reforms to be explored and appropriate theories and methodologies are needed for studying phenomena related to mathematics curriculum reforms. Drawing on the findings and lessons learned from the study, this lecture intends to look into ongoing curriculum reform in school mathematics.

In this paper, we focus on the concept of “alignment” that refers in general to “an agreement or a match between two categories” (Squires, 2009). As for the use in the curriculum study, the term “alignment” typically means a state in which educational goals, instruction, and evaluation are working, or not, in a consistent and functional manner. In particular, from the perspective on three levels of school mathematics curriculum, *intended curriculum*, *implemented curriculum*, and *attained curriculum*, (Valverde et al., 2002), which is the basic framework for TIMSS curriculum analysis (Schmidt, et al., 1997; Mullis & Martin, 2017), we examine theoretically and empirically the alignment of the new curriculum in the case of Japanese curriculum reform toward a competency-based curriculum.

The goals of teaching mathematics described in the new curriculum standards focus on the functions of mathematical perspectives and mathematical thinking in learning mathematics and emphasize an incorporating a cycle of mathematical activities into classroom instruction (Ministry of Education, Culture, Sports, Science and Technology, 2017a, 2017b, 2018). As Kilpatrick (2023) points out the complexity of three levels of school mathematics curriculum, “A single research study can address only some aspects of those curriculum levels” (p.34), we intend to consider comprehensively the findings from a series of studies conducted in the project of Alignment of School Mathematics Curriculum, to grasp the current state and issues of school mathematics curriculum alignment, and identify future research topics and suggestions for classroom practice.

Conceptualizing and Defining School Mathematics Curriculum

Difficulties in Defining School Mathematics Curriculum

There are a number of challenges in studying curriculum reform internationally. One of the most outstanding difficulties exists in defining the term ‘curriculum’ itself. Different countries or parts of the world develop or draw on different meanings of curriculum. It is also important to observe that, as much as mathematics education scholars and practitioners develop their own concepts or definitions, equally there are advances in general education or curriculum studies that are drawn on and influence these meanings. The difficulty in defining school mathematics curriculum partly derived from the complexity and multi-facet nature as an endeavor with a variety of stakeholders (Shimizu & Vithal, 2023).

It is evident from the literature that the term ‘curriculum’ is used with many different meanings and scholars have noted that it seems almost impossible to give a universally acceptable definition of ‘curriculum’. For example, in the US it often means a textbook series, as is manifested in the study on the impact of a Standard-based curriculum on teacher’s classroom instruction and student learning (e.g., Cai et al., 2014). In the UK, on the other hand, the term means the set of experiences a child has in school classrooms (Burkhardt, 2014), while the term ‘curriculum’ is used to mean national documents such as standards or frameworks that describe goals and content of school education in some countries. OECD (2020) points out, “no universal consensus exists on a definition of curriculum, and it is often a contested concept” (p.11), to argue that a more nuanced definition is necessary.

From a general education perspective, Ornstein and Hunkins argue that, “the way we define curriculum reflects our approach to it” (2018, p. 26). For example, for them, curriculum defined as a plan for achieving educational goals may be aligned to a behavioural or system approach to curriculum, while curriculum defined as comprising the experiences learners have in school reflect humanistic approaches.

In some East Asian countries, given the centralized system of education, the meaning of the term curriculum can be different from those in the Western countries for the curriculum documents are prepared and authorized at the national level and there is a process of screening and authorization of textbooks by the national government (Usiskin & Willmore, 2008).

A Framework for Studying Alignment of School Mathematics Curriculum

It can be noted from the literature that how curriculum is defined is related to its purpose. An enduring and widely deployed conceptualization of curriculum is the one that has been used in the Trends in International Mathematics and Science Study (TIMSS) over several decades. The TIMSS Curriculum Model for 2019 continues to define curriculum in terms of three levels: the intended, implemented and attained curriculum – as its core major organising conceptualisation (Mullis et al., 2016) since its earliest studies. Summary descriptions for each level are provided below (Shimizu & Vithal, 2023, pp.14-15).

The *intended curriculum* can be variously described as the planned, official curriculum at an educational system level, for example at a national, district or state level. It is the overt, explicit curriculum in the form of written documents, texts and materials chosen to support the intended curriculum agenda. The intended curriculum is the formal curriculum which typically specifies what the relevant educational authority (and society) expect students will learn in terms of knowledge, skills, values and attitudes. Embodied in curriculum frameworks, guides, regulatory documents, policies or standards are the various curriculum components – goals, content, pedagogy, materials, assessments, examinations and other aspects. It may be the intended institutional curriculum.

The *implemented curriculum* is the actual curriculum delivered at the level of the school and the classroom. It is also defined as the curriculum-in-action or the enacted curriculum, through which the teacher in the classroom interprets and translates the intended curriculum in practice within a school. While the intended curriculum is contextualised with reference to broader national, social and educational contexts, the implemented curriculum is realised with reference to the context of the school and classroom, and the teachers' knowledge and experience. Also defined as the taught curriculum, it is the actual teaching and learning activities which take place through interactions between teachers and students – the teaching methodologies and resources through which the content and assessments are delivered in practice.

The third level, the *attained curriculum*, can also be referred to as the received curriculum or the curriculum as experienced by the students and manifested in their achievement and attitudes. The attained curriculum indicates the knowledge, skills, values and attitudes acquired by learners as a result of teaching and learning through different means and materials and demonstrated in practice. It has been documented and shown that there can be great disjuncture between each level of the intended, implemented and attained curriculum. The power of this definition is that, while a government or educational authority, in very unequal societies, can claim to offer the same curriculum when in fact they are referring only to the intended official curriculum, the implemented and/or the attained curriculum can reveal deep inequalities given by different resources, etc.

It is evident from the above that, at each curriculum level (intended, implemented and attained), attention can be paid to a range of different curriculum components (goals, content, teaching approaches, materials and assessment). For the purposes of this study on alignment of mathematics curriculum, these two dimensions – curriculum levels and curriculum components – offer a means for the core conceptualising of the notion of curriculum.

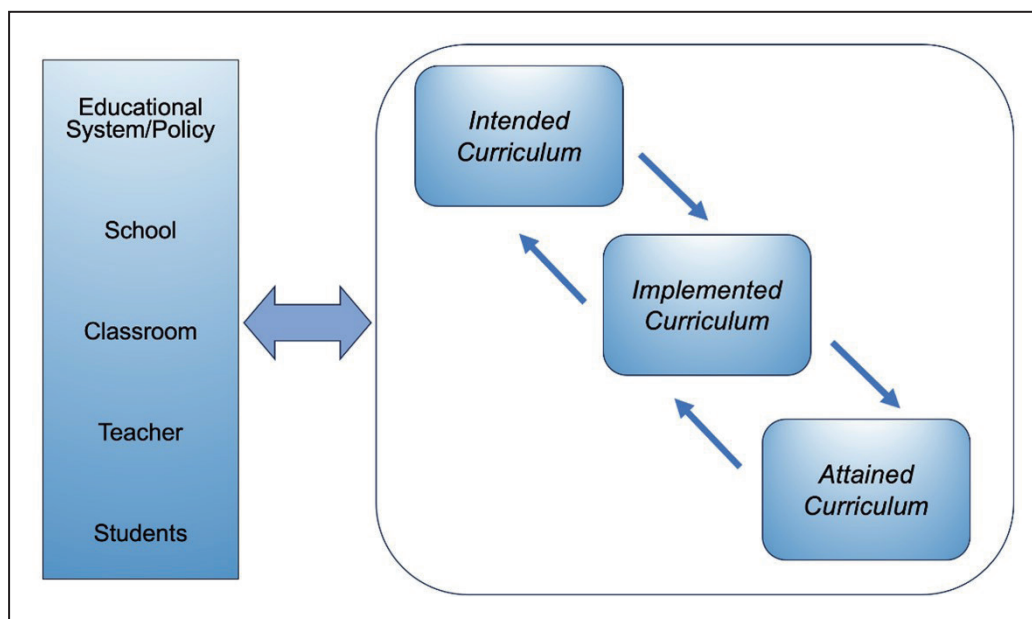


Figure 1. A framework for understanding alignment of curriculum

In addition, in order to consider the alignment of school mathematics curricula, especially during the transition period, we first focused on the relationship between “intended” and “implemented” based on the purpose of the curriculum standards and considered the following three perspectives: That is, (a) positioning the mathematical process as a standard in the curriculum, (b) focusing on new textbooks, and (c) focusing on stakeholders related to the three levels. These are due to the fact that the goals and content of mathematics, that are presented from a new perspective on competency, have been set with particular emphasis on the role of mathematical perspectives and thinking, with a focus on mathematical activities as the process of learning mathematics. In particular, an attention is being paid to the role of textbooks as “potentially implemented curricula” positioned halfway between intended and implemented, and to how the above subject goals and content are specifically expressed.

Also relevant for conceptualising curriculum in a study on mathematics curriculum reforms is a definition of curriculum as process and as product. The term ‘curriculum’ can be used both as a product and as a process. A curriculum is a product: a set of instructional guidelines and materials for students’ acquisition of certain culturally valued knowledge and skill. A curriculum can also be viewed as a process. In this sense, the curriculum is not a physical thing, like textbooks, but rather the interaction of teacher, students and knowledge (Cai & Howson, 2013). Both of these – curriculum as process and as product – can be applied to each level of the intended, implemented and attained curriculum. For example, curriculum as process can refer to processes in curriculum development – at the intended level, this could refer to mathematics curriculum policy-making processes.

Recently, OECD proposed a new approach to study curriculum in the Education 2030 project, with a message to all stakeholders to understand that a curriculum is a living and evolving tool. In recognition of that curriculum viewed broadly is both dynamic and interactive complexity, the Education 2030 definition of curriculum aims to be inclusive (covering both the formal curriculum and the hidden curriculum), multilayered (scoping different aspects of curriculum and covering both mandatory and non-mandatory curriculum content), and dynamic, holistic and multidirectional (taking an ecosystem approach to curriculum rather than a linear, industrial model) (OECD, 2020). Then, in

addition to the three layers of “traditional” curriculum model, related conceptualization such as the expected curriculum, the negotiated curriculum, the perceived curriculum, the experienced curriculum, and the evaluated curriculum (OECD, 2020, p.17).

**The expected curriculum refers to the expectations and beliefs of stakeholders (including parents, students, teachers, school leaders, employers and other community members) around how an intended curriculum should look, what students are to learn, what they should be able to become and how they should act. The negotiated curriculum refers to the process of negotiation between policy makers, teachers, and students before the curriculum is implemented and taught.*

**The perceived curriculum refers to how school readers and teachers perceive the curriculum and what they interpret or understand from the curriculum. The experienced curriculum is the curriculum as perceived and actually experienced by students, while the assessed curriculum refers to the learnings identified through assessment practices designed to capture a subset of student learning. Although these two curricula should be similar, what is experienced and what is assessed are not always identical.*

**The evaluated curriculum constitutes the overall assessment design and implemented of the curriculum, including monitoring and evaluation. It occurs over time and involves multiple stakeholders, including administrators, teachers, parents and others in a process of understanding the life cycle of the curriculum design process. The various dimensions of curriculum are not always in alignment, leading to disconnects and unintended consequences.*

Conceptualization of curriculum by the Education 2030 project focuses on the dynamic, holistic, and multidirectional nature of curriculum and with the chronosystem approach we can see the change of curriculum over time. Thus, the category such as “perceived curriculum” and “experienced curriculum” can provide opportunities for the researcher to examine the viewpoints of stakeholders from their perspectives on the different levels of curriculum.

The Japanese Context:

From the Content to Competencies: A Shift in the Focus of School Mathematics Curriculum

A major element in all school mathematics curriculum reforms is the concern about the treatment of mathematical content within the standards. Such questions as “what is the appropriate content to be taught in school mathematics?”, “how is it represented at all levels and for different students?”, and “what is its original purpose of teaching particular content?”, and so on, are asked in any mathematics curriculum reform effort. In ICMI Study24, several shifts in content aspect of school mathematics curriculum reforms are identified such as moving toward competencies-oriented curricula, a focus on the role of mathematical literacy and on applications and mathematical modelling, as well as a new emphasis on introducing computational thinking or algorithmic thinking (Shimizu & Vithal, 2023).

The recent Mathematics Curriculum Document Analysis (MCDA) study by OECD reports a diversity in the extents to which issues related to the development of mathematics literacy are now being included in national curriculum standards and in the textbooks (Schmidt, et al., 2022). In particular, mathematics literacy includes knowledge of statistical procedures and statistical reasoning that is increasingly more important in making informed decisions related to both the world of work as well as to personal decisions about health, family finances, schooling options, and filing tax returns, but also societal issues such as climate change, inflation rates, income tax policies and country budgets (Schmidt, et al., 2022). MCDA project, as a part of the OECD Future of Education and Skills

2030 (Education 2030) project, explored and compared how much and in what ways has the mathematics curriculum changed in terms of its coverage of formal mathematics over the first 8 grades of schooling, and the inclusion of topics that were not typically taught in the mathematics curriculum some 25 years ago as examined in the 1995TIMSS curriculum study (Schmidt et al., 1997), and the extent to which issues related to the development of mathematics literacy are now being included in the national curriculum standards and in the textbooks used by the teachers.

In Japan, one of the participating countries in MCDA study, the revised national curriculum standards have been released for the implementation of a “competency-based” curriculum at elementary and lower secondary schools (Ministry of Education, Culture, Sports, Science and Technology, 2017) and upper secondary schools (Ministry of Education, Culture, Sports, Science and Technology, 2018). The new national curriculum standards were implemented from 2020 and later. In the revised national curriculum standards, students’ competencies to be fostered through teaching and learning of mathematics were prioritized in the description of goals of each school subject, compared to the previous standards that focused on the content to be taught at each grade level. A “content-oriented curriculum was revised as a more “competency-based” curriculum.

In the revised national curriculum standards, a particular emphasis is given on the role of “mathematical perspectives” (“discipline-based epistemology”), or the ways of seeing and thinking things mathematically, expected to work during learning mathematics in the classroom and beyond. Also, it has been emphasized the importance of mathematical activities in the real world contexts and within mathematical contexts as the process of learning mathematics, including finding and posing mathematical problems, devising a plan to tackle to it, and solve mathematically formulated tasks, and looking back the process engaged to find a new problem to solve.

Alignment of School Mathematics Curriculum: Current Situation and Challenges

Research Items in the Project of Alignment of School Mathematics Curriculum

In the project of Alignment of School Mathematics Curriculum, based on the framework for understanding alignment of curriculum (Figure 1.), the following research items have been selected for addressing corresponding research questions.

Developing a framework for examining alignment across the educational system

The framework of the study presented in Fig.1 includes educational system and policy on the left-hand side as influence on the three levels dynamically. It is necessary to establish a framework for understanding curriculum alignment across layers governed by the various organizations consist of Japanese education system. In particular, it is necessary to consider the intended alignment of curriculum between the national, regional level, and school level. Specifically, we aimed to identify the process by which the national curriculum standards can be disseminated through various pathway to local government boards of education, individual schools, and the teaching and learning processes in individual classrooms.

Analysis of mathematics textbook with a focus on the new emphases

The role of mathematics textbooks, characterized as “potentially implemented curriculum”, is crucial to the alignment between “intended” and “implemented” in Japanese educational system, in particular. Thus, we intend to capture to what extent mathematics textbooks include the new emphasis on the pages, if any. To that aim, we have developed a coding system and use it to analyze

specifically the unit of “Proportional and Inverse Proportional” in the area of “Change and Relations” in all the six different textbooks commercially available for elementary schools. The focus of the analysis was on the tasks and expected students’ activities. In particular, to what extent each textbook incorporates illustration/description of “mathematical perspectives”, and the introduction of mathematical activities on the pages.

Approach to the implemented curriculum: Teachers’ perception and their classroom teaching

In order to grasp the relationship between the three levels of curriculum, it is necessary to explore the alignment between the “intended” and “implemented” of the mathematics curriculum. We have designed a series of lesson study meeting with teachers to analyze the lesson plans, records of lesson and post lesson meetings, transcripts of teacher interviews, and revised lesson plans after lesson studies for example lessons from elementary and junior high schools. In particular, we focused on how teachers perceive and interpret the written national documents as intended curriculum for designing and implementing lessons. We intended to collect different qualitative data from the context of the school and classroom involved in “implemented,” and analyze and interpret it using a consistent perspective and method.

Examining the results of the national academic achievement test

Among the three levels of the curriculum, “attained curriculum” is related directly to student achievement and attitudes. It can be approached from a variety of perspectives, including understanding students’ learning situations, analyzing learning behavior in the classroom, and teachers’ evaluations of students, in addition to their achievement in mathematics assessment. In this study, we began by analyzing the characteristics of the questions and results of the National Survey of Academic Achievement and Learning Status, that is conducted by the Ministry of Education annually with all the 6th and 9th graders in mathematics and Japanese. The results of this particular survey provide indication of attained curriculum with a focus on the new highlights and emphases in national curriculum standards.

A Framework for Analyzing Alignment Across the Educational Systems

Curriculum reform in Japan is centered on revising national curriculum standards, the National Course of Study. This requires consideration of the alignment of intended curricula between the national and local governments. Meanwhile, in Japan, a system for alignment of intended curricula between the national and local governments has been established based on laws and regulations.

Miyazaki (2021, 2022) has proposed a framework for understanding curriculum alignment across layers overseen by the various organizations that consist of the entire education system, and has compiled a list of the mechanisms that have been established in Japan based on laws and regulations (Figure 2.). This figure visualizes the mechanisms in Japan that ensure the alignment between the intended curricula of the country and regions based on the particular laws and regulations. The mechanism itself is the subject of research, and it has been pointed out that it will be possible to identify areas for improvement in the specific content of the mechanisms that connect the country and regions, particularly in terms of instruction in each school subject, and to propose measures that are in line with the actual circumstances of subjects.

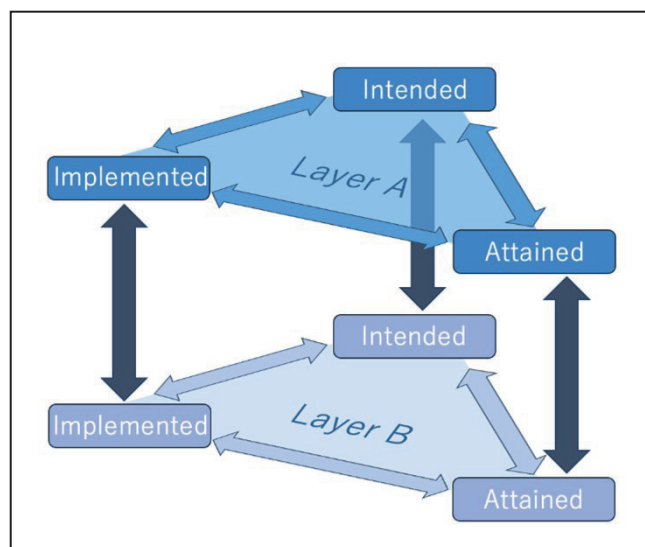


Figure 2. Bidirectional curriculum alignment at two organizational layers in an educational system (Miyazaki, 2020, p. 340)

Furthermore, Miyazaki et al. (2023) analyzed how information on the changes in educational policy can be disseminated to regional boards of education and school in various pathways. In this study, the pathway was selected to be examined the alignment between national level and regional level through a text analysis of materials prepared by the curriculum specialist in mathematics for the meeting with supervisors of each regional boards of education during the period from November 2011 to June 2021. In Japanese system, a curriculum specialist in each school subject is responsible for providing information and materials for implementing the revised national curriculum standards. He or she may give workshops or lectures on the revised national curriculum standards in the curriculum reform.

The analysis reveals the followings. (1) Over a period of more than three and a half years, from November 2012 to Jun 2016, the Ministry of Education disseminate key messages that the competency for thinking mathematically has been set as the goals of the subject of mathematics, based on the trends and values of mathematics in academia and the real world, and the emphasis on mathematical activities as the process of learning mathematics before the revision of curriculum standards (Ministry of Education, Culture, Sports, Science and Technology, 2017a, 2017b). (2) Over a period of more than four years, from June 2016 to November 2020, the Ministry of Education addressed the assessment of competency and the focus of the information has been changed from what is to be assessed to how to assessed and providing example of assessment of competency.

Analyzing Textbooks as Potentially Implemented Curriculum

Textbooks and other organized resource materials can be characterized as a potentially implemented curriculum (Valverde et al., 2002, p.13) that is located between “intended” and “implemented” in the school mathematics curriculum (Figure 3.). In the context of Japanese education, which includes a process of screening and authorization of textbooks by the national government, textbooks may be “closer” to the intended curriculum than other educational systems.

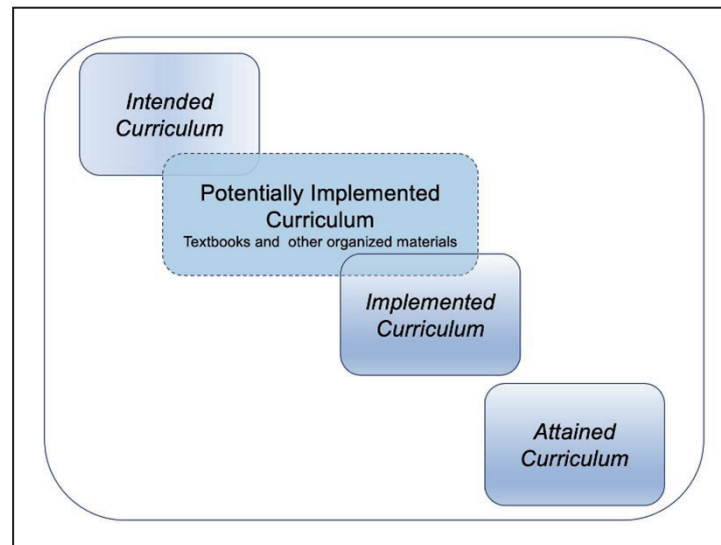


Figure 3. Textbooks and Potentially Implemented Curriculum

To examine the alignment between intended curriculum and implemented curriculum, we have developed a coding system for analyzing textbooks of the unit “Proportional and Inverse Proportional” in the content area of “Change and Relationship” in all the six different elementary school textbooks commercially available in Japan (Shimizu et al., 2021; Shimizu, et al., 2025). We employed the method of textbook analysis as reported in the study of curriculum of Valverde et al. (2002). This method was developed for the Survey of Mathematics and Science Opportunities (SMSO) by a group of researchers from six countries in time for the Third International Mathematics and Science Study (TIMSS) carried out by The International Association for the Evaluation of Educational Achievement (IEA). Their analysis focused on the content in the textbooks that reflected the curriculum of the countries participating in the study.

As the current textbook analysis examined alignment between the intended and the implemented aspects in the school mathematics curriculum, codes were devised not to systematically cover all mathematical content of the textbook but rather to capture items related to clearly stated objectives in the current version of national curriculum standards. The coding system was consisted of categories of sub-units of mathematical perspectives/thinking and mathematical activities, which are explicitly stated as subject objectives in the new national curriculum standards.

The coding of six textbooks series was carried out by the same pair of coders who worked separately and then checked each other’s results. We used the proportion of agreement for each block and Cohen’s Kappa coefficient as the reliability indicator to evaluate the degree of agreement of all coding. If there were discrepancies in coding, the coders consulted to reach a consensus decision. Both coders were graduate school students without actual teaching experience. In carrying out the analysis, the number were recorded of codes assigned to the “blocks” in the textbooks of mathematical perspectives/thinking and mathematical activities.

Table 1 shows codes for mathematical perspectives/thinking and Table 2 shows codes for mathematical activities. Textbook pages are divided into a collection of blocks to be coded,

Table 1. Codes for Mathematical Perspectives/Thinking

Code	Items in mathematical perspectives /thinking
a ₁	When investigating a single quantity, determining another quantity that is deeply related to it.
a ₂	Focusing on the relationship between two quantities
b ₁	Investigating the relationship between two quantities using tables, formulae and graphs.
b ₂	Identifying characteristics of change or correspondence in a functional relationship.
c	Using proportion/inverse proportion, to capture, examine, and express a concrete phenomenon.

Table 2. Codes for Mathematical Activities

Code	Items in mathematical activities
A01	Problems in daily life and social phenomena
A02	Problems from mathematical phenomena
A1	Expressing daily or social phenomena using mathematics
A2	Expressing mathematical phenomena using mathematics
B	Moving from a mathematically expressed problem to a focused problem
C	Solving a focused problem
D1	Using the solution to solve problems in daily and social phenomena
D2	Integrating/developing mathematical phenomena from the solution

The analysis revealed that not all aspects of “mathematical perspectives/thinking” as emphasized in the national curriculum standards appeared on the pages of textbooks (Shimizu et al., 2025). As for mathematical perspectives/thinking, the proportion of items codes as b₂ was particularly large for all textbooks, at between 46.8 and 71.1%. This is because at the level of elementary school, the two quantities of a concrete phenomenon are presented beforehand, and children are asked to consider the relationship with the assumption that it is either a proportion or inverse proportion. B₁ was the second most numerous code (between 7.1 and 24.4% of the total codes). The proportion of items coded as a₁ or a₂ was small (some textbooks did not contain a₁). The results reveal that in many cases, the two quantities were given at the beginning of the problem, which means there were few questions and activities that identifies two quantities that varied together. That is, this reflects that the aim of teaching proportion/inverse proportion in elementary school is examining how two quantities in a proportional relationship change. Furthermore, some textbooks did not contain code c. This means that even for material that used daily phenomena, the assumption was that the two quantities observed a proportional relationship and children would solve the problem based on this assumption. In this case, while the publisher might have posed questions and activities regarding concrete phenomena that required the use of proportion/inverse proportion to solve the problem, if the teacher cannot plan a lesson that uses proportion or inverse proportion to solve the problem, children may not be able to develop appropriate awareness.

Also, the results of analysis reveal that textbooks do not cover all types of mathematical activities emphasized in the new national curriculum standards and that there are certain limitations in the types of problem included. The results suggest that teachers need to supplement tasks and activities that are not explicitly provided in the textbooks for achieving the new competency-oriented goals intended by the national curriculum standards.

As the next step, we expanded the textbook analysis for examining the alignment in textbooks to the secondary school level, targeting the units “Function in the form of $y = ax^2$ ” in grade 9 at junior high school and “Quadratic Functions” in grade 10 at senior high school Mathematics I (Iwata et al., 2022; Nishimura et al., 2023). The results of the examination using the analytical unit set in this study suggested that, although there are exceptions, high school textbooks have fewer blocks of “questions” and “activities” compared to those for junior high school, and the reflection and implementation of “intended curriculum” is left to the teacher.

Furthermore, the authors analyzed the chapter on “Quadratic Functions” in high school Mathematics I textbooks. The analysis reveals that there are differences between textbooks in “mathematical perspectives/thinking,” with particularly large differences in the presence or absence of descriptions regarding “paying attention to the mutual relationships between function tables, equations, and graphs.” In terms of “mathematical activities,” the study found a high proportion of textbooks that “started with solving a focused problem,” and that there were very few activities that involved mathematically considering everyday phenomena or social events.

In order to achieve the goals of the new curriculum standards, it is necessary for teachers to incorporate into lessons the processes of problem finding, mathematization of phenomena, and integration and development that are typically not found on the pages of textbooks, as well as supplementing lessons with questions and activities that allow students to become aware of mathematical perspectives and ways of thinking that are important in problem solving. On the other hand, from the perspective of alignment, it is also important to propose how to supplement and realize through instruction what is difficult to express on the pages of textbooks.

Understanding the Characteristics of the Implemented Curriculum: Case Studies

Analysing Teachers’ Interpretation of the Intended Curriculum

Given that the alignment between the intended curriculum and implemented curriculum need to be examined with the data of what is actually taught in the classroom. A series of analyses of classroom practice have been conducted in the project. Saito and Kageyama (2022), for example, proposed a perspective to grasp the consistency of alignment through a multidimensional analysis approach. This multidimensional analysis approach involves comprehensively looking at qualitative data from different contexts. They selected word “reasoning” as the use of the word and conducting a text analysis, it became clear that word use differs depending on the curriculum levels, “intended” and “implemented.” The teacher’s unique interpretation and translation are taken into account, and it is proposed that when counterparts from each level can be seen, it is determined that the consistency of the curriculum is maintained.

On the other hand, Ichikawa et al. (2022) clarified the actual state of alignment and conducted a case study to show what adjustments are made to resolve inconsistencies when they exist. In this study, a research lesson in junior high school mathematics was analyzed. It was found that the perspective/way of seeing similarity was reflected in all aspects, from concept to plan to actual practice, but the awareness of how to apply this perspective/way of thinking was not reflected in all aspects. In addition, through a research conference, it was confirmed that adjustments were being

made between the “intended curriculum” and the “implemented curriculum” in terms of mathematical perspectives/ways of thinking and how they are applied in terms of concept. This study confirmed that in lesson study meetings, through research conferences, adjustments are made between “intended” and “implemented” in terms of the ideological aspects of mathematical perspectives and ways of thinking and how to apply them. In this sense, it has become clear that lesson study plays the role of a bridge between “intended” and “implemented.”

Furthermore, Seino et al. (2023) used case studies to clarify how teachers who are not mathematics specialists make adjustments to resolve the inconsistency between “intended” and “implemented” through lesson study. Specifically, they analyzed a teacher at a school that had been conducting intra-school research focusing on mathematics for one year, and analyzed the lesson plan review meeting, research lesson, and research conference that were held before the research lesson, as well as interviewing the teacher and analyzing the records. As a result of the analysis, it was confirmed that the teacher constantly recalled his/her own lessons to date, enriching his/her view of teaching materials and teaching, while filling the gap between “intended “ and “implemented curriculum.”

An Approach to the Perceived Curriculum

To explore how teachers plan and design a series of lessons in accordance with the new emphasis of national course of study, two experienced teachers were invited to conduct research lessons in the form of lesson study (Shimizu, 2024). One of the experienced teachers, Mr. T has 25 years of teaching experience. He taught 5th grade classroom on the topic of “geometrical figure and angles” that is intended to be treated differently (in the different content areas) between the former and the new National Curriculum Standards. The research lesson and the post-lesson discussion were video-taped. Data sources included a plan of the entire teaching unit with the detailed lesson plans for the six lessons prepared, the verbatim transcriptions of video-recorded lesson, the record of the post-lesson discussion, and the fieldnotes of researchers.

Mr. T developed a teaching unit on “the sum of the angles of a polygon” that consists of six lessons with a set of detailed lesson plans. He explicitly described his interpretations of the new national curriculum standards related to the area of geometry in his lesson plans. In particular, he referred to the emphasis on the mathematical competencies in the new national curriculum standards and focused on fostering students’ logical thinking.

In the lesson observed, he aimed to invite students to find that the sum of four angles of a quadrilateral is 360 degrees, if we can use the fact that the sum of three angles of a triangle is 180 degrees.

Table 3. The Plan of Unit: The Sum of the Angles of a Polygon (5th grade)

Lesson	Topic
L1	Finding the sum of three angles of a triangle inductively by using unit squares.
L2	Understanding the sum of three angles of a triangle is 180 degrees. (How about a square?)
L3	Confirming that the sum of four angles of a quadrilateral is 360 degrees.
L4	Explaining that the sum of four angles of a quadrilateral is 360 degrees, based on that the sum of three angles of a triangle.
L5	Finding the sum of the angles of a polygon
L6	Making a nice pattern by the tessellation with quadrilaterals

We observed and analyzed the third lesson where students confirmed that the sum of four angles of a quadrilateral is 360 degrees. In his lesson plan, Mr. T described the goal of lesson as follows: Students can confirm it logically that the sum of four angles of a quadrilateral is 360 degrees, based on the fact that the sum of three angles of a triangle is 180 degrees. He explained that he was aware of the description of teaching geometry in the supplementary curriculum document from the Ministry of Education: “It is crucial that once the students acquire a new mathematical perspective, then they apply the perspective to the figures they learned so far with the purpose of integrating them” (p.16). For this reason, Mr. T intentionally plan to selected and order the figures to be presented in his classroom: starting from square and rectangle and then moved to parallelogram, followed by rhombus and trapezoid. Finally, a general quadrilateral was presented.

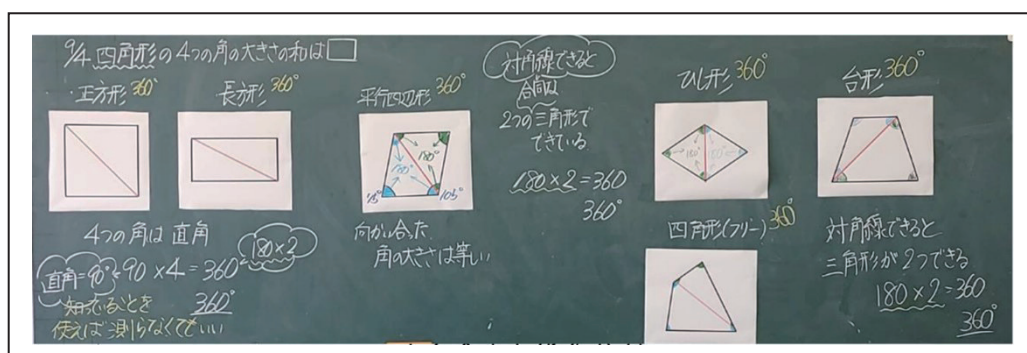


Figure 4. Quadrilaterals Presented and Discussed on the Chalkboard

In his reflection after the lesson study meeting, he wrote as follows.

In the classroom, I found that some students were trying to measure the angles with a protractor anytime they saw a new quadrilateral such as rhombus and trapezoid. In this lesson, I intended to facilitate my students to look at the figures they know with a new perspective when they acquired it. Specifically, through learning that any quadrilateral can be divided into two triangles and that the sum of four angles of a quadrilateral can be found by 180×2 , I aimed to develop their competencies for integrating what they found.

This might be caused by my own awareness of finding the sum of the angles by using the measure of each angle. The unit needs to be treated as the area of geometrical figure, not the area of measurement.

The results of analysis reveal that the teacher interpreted the curriculum standards from his own perspectives on school mathematics. If we call this as the perceived curriculum, it can be manifested in the form of lesson plans and the instructional materials into his classroom. He “customizes” what is written in the national curriculum document in accordance with the goal they set and the reality of their students in the classroom (Shimizu, 2024). From a research perspective, a focus on the perceived curriculum can provide an opportunity to reveal how school leaders and teachers perceive the curriculum and what they interpret or understand from the curriculum before their implemented.

The case study of Mr. T suggests that a flexibility is given to each school and teachers who interpret and translate intended curriculum to design student activities in their classroom as implemented curriculum, while school mathematics curriculum reforms are undergoing with a shift toward a competency-based curriculum.

Discussion

In this chapter, we discuss the findings of studies mentioned in the previous chapter to argue challenges and opportunities in the study of alignment of school mathematics curriculum and to identify the directions of future research emerged from the current study. It should be noted that the school mathematics curriculum is deeply contextualized in the social situation and multi-dimensional in nature. As is mentioned in the previous broader review, it may not possible to capture well the issues related curriculum in a single study, framework, and trends (Li & Lappan, 2014, pp.6-9). We need to consider aspects of alignment in a transitional period of curriculum reforms comprehensively with an attention to all the three levels of curriculum with the viewpoint of stakeholders.

Forming and Disseminating “Intended Curriculum”

In Japan, the curriculum reform has been conducted through the revision of the national curriculum standards in the span of roughly ten years. Before and after the revision, attention is given to how “intended” curriculum is formed through discussions and the official “reports” by the Central Council for Education, among others.

From a mathematics educator’s perspective, there is a “distance” between the policy talks around the general policy and curriculum issues related mathematics education. Meanwhile, in addition to the national curriculum standards for mathematics, the publication of “supplementary materials” to the revised national curriculum standards provides teachers with more specific information on each subject. In addition, curriculum specialists at the Ministry of Education, who are responsible for each school subject, explain the purpose and emphasis of the national curriculum standards at forums or seminars such as the “National Meeting of Directors of Supervisors” and “the intended” are communicated to schools through local government boards of education.

In a national education system, curriculum alignment works on the intended, implemented, and attained curriculum that are hierarchically governed by the national and local governments, schools, and other organizations. In particular, for an organization in educational administration to adjust the three difference levels of curriculum, the intended curriculum serves as the fundamental starting point. The text mining analysis by Miyazaki et al. (2023) provides empirical data on this formation and diffusion process. It is expected that the analysis will be extended further to identify the qualitative changes in the content of the intended curriculum transmitted from the country to the region and the period required. Although this result focuses on a single subject, it shows the possibility of broadening the scope of analysis and implementing educational policies in accordance with various subjects.

Bosch et al. (2023) point out that research in mathematics education tend to focus on smaller units of analysis like the teaching of a given topic or recurrent student difficulties in learning. Although analyzing curriculum over a long term requires specific research tools and those which will allow a more neutral stance by researchers who are variously involved in the reform. The challenge of a long time-scale of reforms to include the political sensitivity of reform processes at each level and difficulties of obtaining data on the inner workings and dynamics of curriculum committees which could contribute to greater understanding of reforms It appears that there is wide-spread practice of curriculum reform initiatives nationally and regionally across the world but with limited in-depth studies at macro or policy levels to inform evidence-based improvements over time and across contexts.

The Role of Textbooks in the Alignment of Curriculum

Textbooks can be “potentially implemented curriculum” that lies between “intended” and “implemented” (Valverde et al., 2002). In the case of Japan, with a screening and authorization of textbooks by the national government, the challenge is to focus on the role of textbooks and understand how the goals and content set out in the curriculum standards are reflected in textbooks. For example, it would be possible to analyze the extent to which activities envisaged from the goals as “mathematical activities” are presented in some form on the pages in textbooks. The role of textbooks in the alignment is crucial.

Shimizu et al. (2021, 2025) analyzed mathematics textbooks and pointed out that among the categories set for “mathematical perspectives/thinking,” “finding characteristics of change and correspondence in proportional and inverse proportional relationships” accounted for more than half of the total. This way of looking at things and ways of thinking is used when considering changes and correspondences between two quantities that are in proportional or inverse proportional relationships, and when expressing them in tables, equations, and graphs, and it has been found that this is the mathematical way of looking at things and ways of thinking that is primarily used in the “proportional and inverse proportional” unit in elementary school mathematics.

On the other hand, when it comes to “mathematical activities,” it became clear that more than three-quarters of all relevant activities were “solving from a focused problem.” Few of these activities were set up in situations where direct or inverse proportional relations were used to solve problems in concrete phenomena, and many were set up with the intended of acquiring the knowledge and skills necessary to solve problems, such as activities to check whether something is proportional or to express the relationship between two quantities in an equation, etc.

Furthermore, it has become clear that not all of the component of “mathematical perspectives and ways of thinking” and “mathematical activities” are reflected in textbooks. This suggests that, in order for teachers to enable students to solve problems in a real world situation, rather than simply mastering the content shown in the textbook, it is necessary to supplement activities that involve the process of discovering problems, the process of focusing on problems in concrete phenomena, and the process of returning the problem to the original phenomenon after solving it, in order to enable students to solve problems in concrete phenomena. The role of textbooks needs to be understood as being supplemented by teachers, who are primarily responsible for “implemented,” and this point needs to be further examined at the level of “implemented.”

Teachers as Key Stakeholders: A Focus on the Perceived Curriculum

In Japan, teachers interpret the “intended” through the announced curriculum standards, their “explanations,” textbooks, etc., and then move on to “implemented” in the classroom. The degree to which teachers interpret this “intended” will vary, but it is an important task to analyze the interpretation of “intended” by teacher groups, who are also stakeholders.

New foci on school mathematics at the level of intended curriculum such as the function of “mathematical perspectives and ways of thinking” and “mathematics activities” as a basis for learning mathematics need to be discussed by group of teachers. The case studies mentioned in the previous chapter represented that lesson study can provide a bridge between “intended” and “implemented.” Also, for each teacher who participate in lesson study it was revealed that the teacher constantly recalled his/her own lessons to date, enriching his/her view of teaching materials and teaching, while filling the gap between “intended “ and “implemented curriculum.”

For the analysis in the case studies reported, it was crucial to be centered on the analysis of the lesson itself, but should also be complemented by analysis of lesson plans and voices of teachers and students. There is a wide recognition of the role of multiple agents in school mathematics curriculum reforms. However, the critical role of teachers stands out across the findings of the project. Fey (2014) notes on the challenge of implemented phase of curriculum reform as follows.

Choosing a course of action is actually the easy part of any school improvement initiative. The real challenge is translating those intended into everyday classroom reality. The implemented phase of reform requires ‘selling’ the plans to concerned teachers and the wider school community, preparing appropriate instructional and assessment materials, providing professional development for teachers, and sustaining support for innovations (p.329)

Even in themes that did not specifically pose research questions related to teachers, it is evident that teachers have the potential to make or break a mathematics curriculum reform.

Attained Curriculum and Its Impact on Setting “The Next”

Although it is not explicitly mentioned to the attained curriculum in this paper, one of the important issues related to student achievement is the pathway from “intended”, “implemented”, and “attained” curriculum to be examined the study of curriculum alignment.

Furthermore, just as the results of international academic achievement surveys such as TIMSS and PISA are reflected in the “reports” of the Central Council for Education, the results of academic achievement surveys can serve as the basis for the “next” revision of the curriculum. Therefore, various approaches are possible to examine the alignment, from using online-assessments by individual teachers in the classroom to a large-scaled assessment of academic ability surveys conducted by the national and local governments and schools. It is necessary to move forward in the further research to analyze relationship between three levels of curriculum locating “attained” curriculum in the central place among the three.

Concluding Remarks

School mathematics curriculum reforms have been large-scale involving an educational system as a whole, both at a national or regional level in which curriculum standards or framework have been developed, implemented, evaluated, and revised. In some East Asian countries with a centralized system of education, intended curriculum can be disseminated in a systematic “top down” manner, while a flexibility is given to each school and teachers who interpret and translate it into student activities in the classroom as implemented curriculum.

In last decades, given the globalization and internationalization of societies, an advocacy of the new curriculum for education for “knowledge-based society” or preparing users of information technology, among many other things. More recently, with the needs of education for SDGs and with generative AI, for instance, school mathematics curriculum reforms are undergoing with a shift toward more competency-based curriculum. This is a change observed in many countries and education systems around the world, where there is a greater focus than ever before on the relationship between educational goals and the results of educational practice based on those goals (Shimizu & Vithal, 2023). In the context of these trends, it is becoming increasingly important to advance the alignment of school mathematics curriculum within each educational system with unique contexts and cultural background. In this way, it should be noted that teachers are key players in achieving the alignment among the three levels of curriculum.

In this paper, we comprehensively consider the findings of the research conducted to date, grasp the current state of school mathematics curriculum alignment, and organize research topics in alignment research based on a framework for understanding school mathematics curriculum. The issues pointed out in this paper are all interrelated, but it is necessary to prepare the necessary data and analytical methods for each item and conduct further empirical research.

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Plenary Lecture 3

**BRIDGING ACHIEVEMENT-MOTIVATION GAPS:
A SYNERGISTIC APPROACH TO QUALITY MATHEMATICS EDUCATION**

Kai-Lin Yang



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



BRIDGING ACHIEVEMENT-MOTIVATION GAPS: A SYNERGISTIC APPROACH TO QUALITY MATHEMATICS EDUCATION

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*Using Bronfenbrenner's bioecological systems as a guiding framework, this paper analyzes the achievement-motivation gaps and introduces four programs designed to bridge these gaps in the pursuit of quality mathematics education. These programs adopted distinct strategies: initially implemented in either non-formal or formal educational settings, and focused on either the gap between high achievement and low motivation, or the gap reflecting wide variation in achievement and motivation. Over time, these programs expanded to address both gaps across diverse settings, guided by a shared rationale: prevention is prioritized over remediation, continuous improvement over the pursuit of superiority, and equity over equality. At the heart of these programs is the **SMILE** framework, consisting of five co-evolving forces spanning all bioecological levels: task **Structure**, adaptive **Momentum**, practice-research **Interplay**, responsible **Leadership**, and lifelong **Engagement**. Among them, practice-research **Interplay** plays a pivotal role, sustaining the others. Together, these forces form a synergistic approach to improving the quality of mathematics education.*

Keywords: Bronfenbrenner; bioecological; mathematics; motivation

Two Types of Achievement-Motivation Gaps

Students' performance in large international studies, such as the Programme for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS), receives significant public attention. Prior research on the relationship between students' motivational factors — such as confidence, interests, values, and goals (Bandura, 1997; Wentzel & Wigfield, 2009) — and their achievement in mathematics has found a positive correlation at the individual level but a negative correlation at the country level (e.g., Chen & Hastedt, 2022; Shen & Pedulla, 2000). The paradoxical relationship between achievement and motivation became a theme of the East Asian learner paradox, also referred to as the Chinese Learner Paradox (Watkins & Biggs, 1996).

Specifically, according to TIMSS 2015, fourth-grade students from East Asian countries, including Japan, Korea, and Taiwan, performed exceptionally well in mathematics. However, their motivational outcomes, such as mathematical confidence, were significantly lower than those of their Western counterparts from countries such as Australia, Canada, Germany, the United Kingdom, and the United States (see Figure 1). While TIMSS 2023 confirmed a similar pattern, the negative correlation between achievement and confidence appeared to be smaller than in 2015 (see Figure 2). The gap between high achievement and low motivation highlights a hidden concern in high-performing education systems; that is, students may experience higher anxiety, which may result from lower confidence when facing failure, making them more likely to give up easily, as suggested by attribution theory (Graham, 1991).

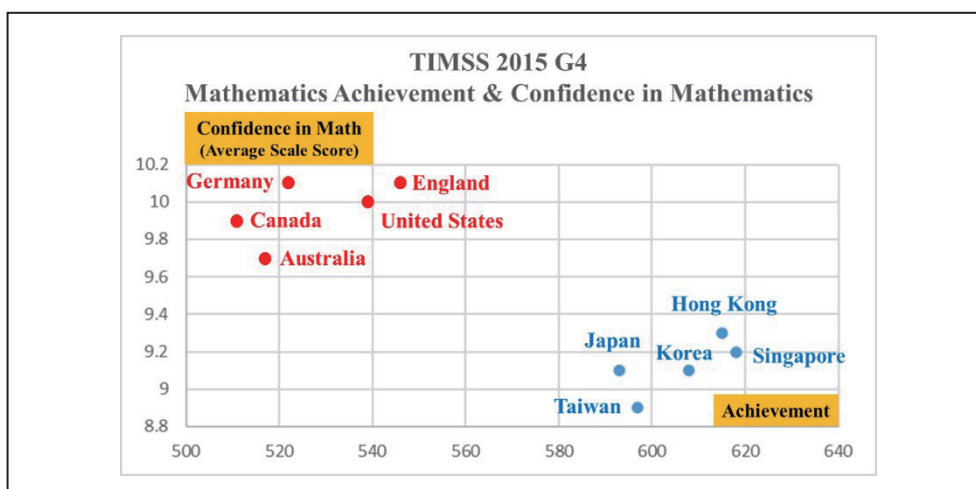


Figure 1. High achievement and low confidence of Grade 4 in TIMSS 2015

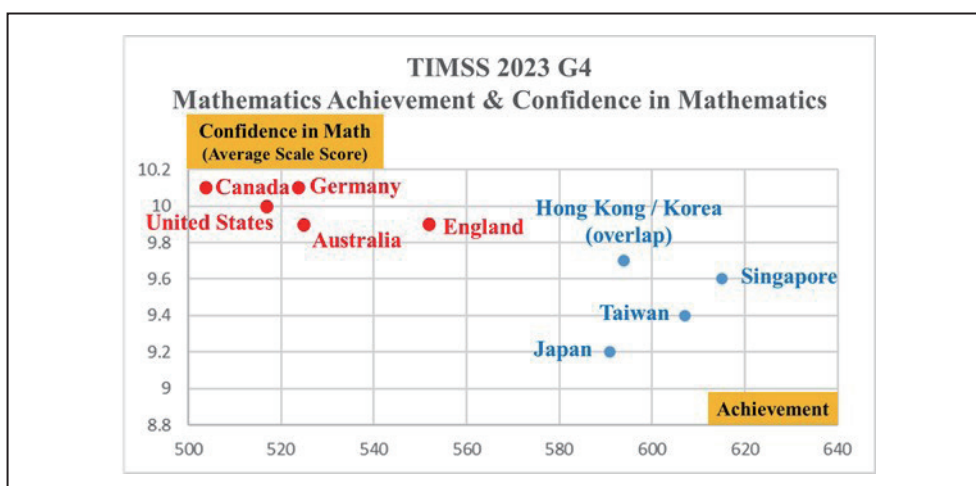


Figure 2. High achievement and low confidence of Grade 4 in TIMSS 2023

In addition to the tension between high achievement and low motivation, disparities within either achievement or motivation themselves also warrant careful examination. According to PISA 2022, East Asian countries ranked among the top performers in mathematics. However, aside from Japan, these countries also exhibited greater within-country variation (as measured by standard deviation) in mathematics literacy scores compared to their Western counterparts, such as Australia, Canada, England, and the United States (see Figure 3). Given that mathematics self-efficacy is closely linked to motivation (Bandura, 1997), we examined self-efficacy results in PISA 2022 and found that, aside from Singapore, East Asian countries reported lower average levels of self-efficacy but greater variation among students relative to Western countries (see Figure 4). The gap in the variation of either achievement or motivation highlights the complex nature of educational disparities in high-performing systems, emphasizing the need to address the inequities in both achievement and motivational factors.

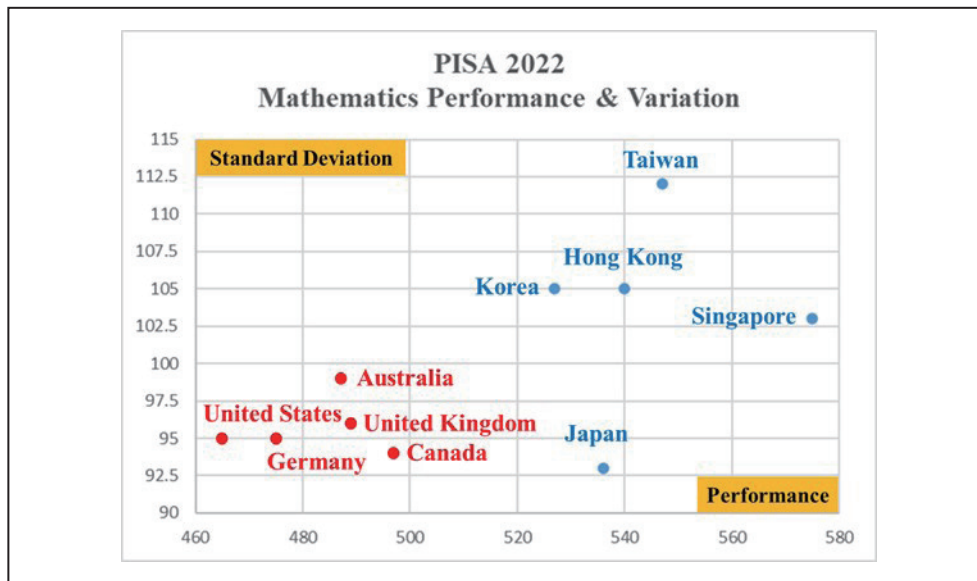


Figure 3. Wide variation in mathematical literacy in PISA 2022

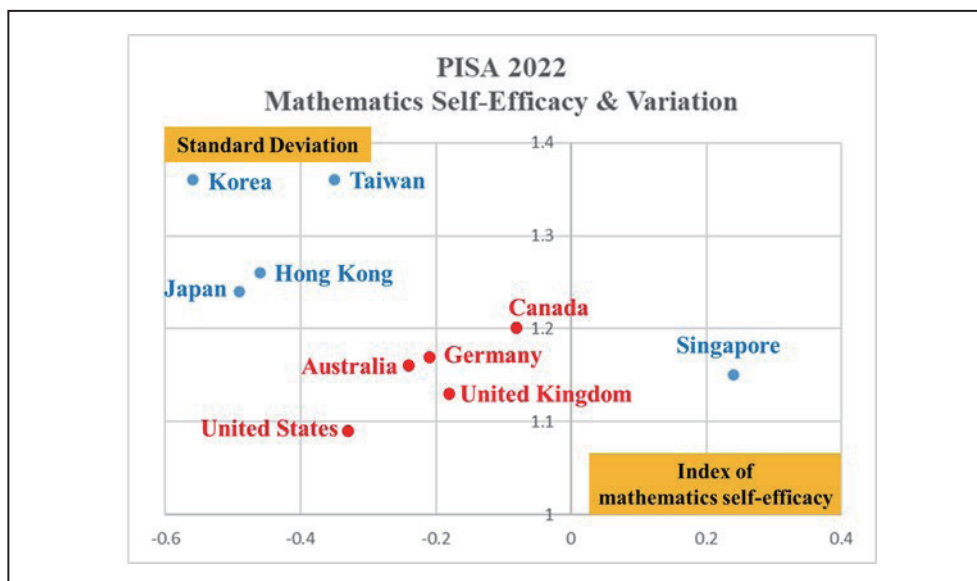


Figure 4. Wide variation in mathematics self-efficacy in PISA 2022

Researchers have provided various explanations for the East Asian learner paradox. These explanations can be organized from the micro-, meso-, exo-, macro- to chrono-system levels, with the developing person at the core, based on Bronfenbrenner's bioecological systems (Bronfenbrenner, 2005). In this context, the developing person refers to the student, who brings prior knowledge, attitudes, motivations, and learning strategies into the educational environment. At the microsystem level, students engage in direct, reciprocal interactions with teachers, learning tasks, and instructional methods. These immediate settings have the most direct influence on students' learning and behavior.

The mesosystem encompasses the interconnections between various microsystems, such as the relationships among peers, parents, school teachers, and cram school instructors. These interactions can significantly influence students' academic experiences and attitudes toward learning. The exosystem includes external settings and decisions made by researchers, curriculum developers, and policymakers, that indirectly influence the student. At the macrosystem level, the focus shifts to the broader cultural and societal values and norms that frame students' educational experiences. This includes prevailing attitudes toward education, competition, or the role of non-formal education systems. Finally, the chronosystem refers to temporal dimensions of development, such as shifts in educational policy or the advancement of digital technologies that affect learning environments over time.

For example, in high-performing East Asian countries, most students use metacognitive strategies combined with memorization to learn mathematics. At the individual level, students who reported using metacognitive strategies alongside either memorization or elaboration demonstrated higher mathematics achievement (Wu et al., 2020). This suggests that repeated practice involving nuanced strategies may serve as an important pathway to achievement, although it could potentially diminish learners' interest over time (Deci & Ryan, 1985; Hess & Azuma, 1991), reflecting an influence at the chronosystem level. At the micro- and meso-system levels, while students in high-performing East Asian countries do not spend significantly more time in mathematics lessons during regular school hours (microsystem) compared to their Western counterparts, they invest substantially more time in out-of-school mathematics activities, such as attending cram schools (mesosystem) (Leung, 2006).

At the exosystem level, high-stakes examinations can, on one hand, foster strong extrinsic motivation, but may limit students' opportunities to engage in and enjoy the learning process (Ryan & Weinstein, 2009). On the other hand, at the macrosystem level, cultural values such as the saying "contentedness leads to loss, humility leads to gain" may shape East Asian students' interpretations of psychological constructs such as interest and confidence, in contrast to their Western counterparts (Leung, 2001). Moreover, the high-performing East Asian students tend to give moderate ratings on psychological construct scales. This suggests that Confucian-Heritage Culture may play a role in the East Asian learner paradox. As illustrated in Figures 1 and 2, Taiwan demonstrates high academic achievement alongside relatively low student confidence. However, Taiwanese fourth graders' mathematical confidence has shown improvement between TIMSS 2015 and TIMSS 2023. This positive trend may be partially attributed to various educational initiatives supported by the Taiwan Ministry of Education, which were designed to enhance students' motivation and engagement in mathematics. These initiatives reflect influences at the chronosystem level, indicating how changes over time in policy and practice can impact student development.

Achievement-motivation gaps influenced by socioeconomic status contribute significantly to educational inequities (Lam & Zhou, 2022). Moreover, the intense emphasis on academic success within Confucian cultural contexts may undermine students' intrinsic motivation, with possible long-term implications for their engagement and well-being (Chen, 2023). Two distinct types of achievement-motivation gaps are evident in top-performing East Asian countries, highlighting the challenge of fostering students' intrinsic motivation and engagement in mathematical thinking. The clear contrast between high achievement and low confidence, combined with considerable variation in both achievement and motivational factors, poses a major challenge for mathematics education in Taiwan (Lin et al., 2018).

Rationale for Bridging the Two Types of Achievement-Motivation Gaps

Through the lens of Bronfenbrenner's bioecological systems (Bronfenbrenner, 2005), I propose three key rationales that guide efforts to address the two achievement-motivation gaps: one between high achievement and low motivation, and another reflecting wide variation in students' achievement and motivation. These rationales, including *prevention over remediation*, *continuous improvement over superiority*, and *equity over equality*, underpin educational practices across multiple bioecological levels.

The first rationale, *prevention over remediation*, emphasizes the importance of early support to minimize unnecessary academic struggles. By preparing learners in advance and fostering adaptability, educational systems can reduce frustration and foster resilience. Support at the exosystem level, such as governmental and institutional resource allocation, enables these proactive efforts by empowering teachers and driving change at the mesosystem level. Turning to the micro- and macro-system levels, the rationale of *continuous improvement over superiority* comes into focus. At the microsystem level, preparedness and adaptability, rooted in the first rationale, support learners' motivation and capacity for sustained thinking and learning. At the macrosystem level, these efforts to promote continuous improvement gradually influence cultural values, emerging an attitude that prioritizes personal engagement and perseverance. This second rationale encourages learners to pursue growth based on their own progress, rather than engaging into competitive comparisons, thereby shifting the focus from being the best to becoming better. Over time, these values and practices are reinforced at the chronosystem level, where a sustained focus on self-improvement and early support fosters a learning culture oriented toward long-term transformation.

Within this broader bioecological perspective, the third rationale, *equity over equality*, becomes particularly salient. While equality assumes that fairness is achieved by treating everyone the same, equity acknowledges and responds to the diverse contexts and conditions that shape each learner's development. This rationale underscores the importance of context-sensitive approaches that address structural and situational differences, ensuring that support is tailored to individual needs. Rather than offering uniform assistance, equity calls for differentiated strategies and resources that empower all learners to participate meaningfully and engage with challenges appropriate to their circumstances. Ultimately, this approach promotes co-improvement for the common good.

Together, these three rationales provide a foundation for fostering a more inclusive and motivating educational environment. Through the dynamic interplay of bioecological systems over time, they guide the creation of structures and cultures that prioritize proactive engagement, value individual growth, and strive toward fairness and justice in educational outcomes. The following four programs demonstrate how these guiding rationales can be translated into practice to gradually evolve more equitable and motivating learning environments, particularly in addressing the two types of achievement-motivation gaps.

Four Programs for Bridging the Two Achievement-Motivation Gaps

Program #1: Just Do Math (JDM)

Although policies had been implemented to close the achievement-motivation gaps among students through remedial programs, their effectiveness was still limited (Lin et al., 2013). To bridge the gap between high achievement and low motivation, the Just-Do-Math (JDM) program was initiated in 2014 with the aim of providing high-quality activities for students to engage in learning mathematics in non-formal education settings, serving as a form of shadow education that builds a productive

foundation for students to learn mathematics in formal education (Yang et al., 2022a). The tasks designed in the JDM were conceived as learning activities, specifically mathematics-grounding activities (MGAs), conducted at camps. In a one-day summer camp, three to four MGAs were conducted, with each activity lasting 90 minutes. Students' learning in MGAs was evaluated, and feedback was collected through questionnaires for each MGA. The findings indicated that MGAs significantly improved both students' cognitive engagement (thinking and problem-solving) and their affective engagement (interest and motivation) (Lin et al., 2018). Most MGAs were designed to develop students' conceptual understanding, while some focused on enhancing students' reasoning abilities, such as spatial reasoning (Yang & Chen, 2024).

In 2016, an increasing number of teachers involved in MGAs-for-camp, particularly those working with elementary school students, began to see the potential of integrating MGAs-for-camp into regular classroom teaching, as they had witnessed how students' active thinking was significantly enhanced through participation in various MGAs-for-camp (Chang et al., 2021). From 2017, the JDM program MGAs were redesigned, specifically as MGAs-in-class, to extend their benefits from shadow to formal mathematics education, enabling successful implementation in regular school lessons. From 2021 on, considering the importance of thinking skills, MGAs-in-class were further designed to incorporate various skills emphasized in the PISA 2021 mathematics framework, such as critical thinking, creativity, and systems thinking. By the end of 2024, the JDM program had developed 178 MGAs-for-camp and 204 MGAs-in-class that were in line with the Grade 3 to Grade 9 mathematics curriculum. To date, approximately 250,000 and 140,000 Taiwanese students have participated in MGAs-for-camps and MGAs-in-class, respectively, more than 30,000 teachers have participated in workshops for cultivating designers, leaders or teachers of MGAs-for-camp, and about 6,000 teachers have participated in workshops for cultivating designers, leaders or teachers of MGAs-in-class.

Drawing on our experience, we highlight three key features in the design and implementation of the JDM program: (1) the integration of both cognitive and affective components in the design of MGAs, reflecting a commitment to *prevention over remediation* by proactively supporting students' motivation and learning preparedness; (2) the development of MGAs-in-class, which serve as a bridge between learning activities in non-formal education and regular classroom teaching, promoting *continuous improvement over superiority* by valuing iterative growth in both students and teachers; and (3) the implementation of professional development programs that facilitate collaborative learning among educators, teachers, and students (Yang et al., 2022a), embodying *equity over equality* by creating responsive and inclusive learning communities. As of 2021, however, several challenges remained: (1) MGAs-in-class were less broadly implemented in secondary schools compared to primary schools, largely due to more rigid curricula and high-stakes examination pressures that limited instructional flexibility; (2) teachers were required to attend mandatory workshops for other programs, such as training on technology integration, which left them with limited time or energy to voluntarily participate in the JDM program's weekend workshops; and (3) insufficient extrinsic motivation, such as professional credits or formal recognition, was provided to encourage teachers' sustained participation in these workshops.

Program #2: Quick-Witted Math Showdown Platform (QMS, Extended from JDM)

Reflecting on these challenges led us to a guiding question: How can we enable more students to experience essential mathematics learning, such as MGAs, within the constraints of the existing school environment? This reflection became the impetus for launching a new initiative focused on the development of digital mathematical games, aimed at fostering students' abilities to generate

new mathematical ideas in ways that are scalable and adaptable across school contexts. Recognizing the varied developmental needs across educational stages, we adopted differentiated strategies: a story-based approach for grades 4 to 6 to spark narrative engagement, an MGA-based approach for grades 7 to 9 to preserve the pedagogical integrity of MGAs within more structured digital environments, and a MGA-like approach for grades 10 to 12 to align with more advanced content and address students' declining interest in mathematics learning. This paper specifically introduces the digital MGA-based approach designed for grades 7 to 9, as it most directly reflects the core structure and learning objectives of the original MGAs, while responding to the implementation challenges observed in secondary education.

A review study on digital games for promoting mathematical reasoning (Jensen & Skott, 2022) identified five distinct categories: solving tasks, exploring an immersive environment, developing (winner) strategies, experimenting, and designing learning games. The first two categories primarily focus on reinforcing existing mathematical knowledge or applying learned procedures. In contrast, the latter three offer potential for engaging students in the development of mathematical reasoning and conceptual understanding. However, each of these categories also presents specific limitations in actively supporting the evolution of new mathematical understanding through processes of mathematizing the game tasks, that is, transforming game-based experiences into mathematical structures, relationships, or representations.

To further classify digital mathematical games based on the nature of mathematical integration within gameplay, we distinguish among three types: games used solely as rewards, games that support the application of mathematics, and games that promote the construction of mathematical understanding through interaction. In the latter case, games function as dynamic learning environments that actively engage students in developing new mathematical reasoning, skills, and concepts through processes such as experimentation, trial and error, and responsive feedback. To expand the availability of MGA-based games, the Quick-Witted Math Showdown (QMS) Platform has been in development since 2022. As of 2024, this platform has been freely accessible to all students upon login, with the aim of promoting rich mathematical thinking within a game-based environment embedded in a broader adaptive learning system website (<https://adl.edu.tw/HomePage/home/>). The design and evaluation of digital MGA-based games are grounded in an enactivist perspective on mathematics learning, which conceptualizes mathematical understanding as an active, co-evolving process shaped through learners' interaction with meaningful tasks and learning environments. Drawing on this perspective, Yang et al. (2022b) introduced a framework for designing MGAs as three interrelated processes: metaphorizing, scaffolding, and gamifying, and their corresponding task evaluation criteria. These processes and criteria serve as a foundation for guiding the design of digital MGA-based games.

Among the featured games on the platform, "Darting Factors" exemplifies the integration of learner, task and social contexts, offering a carefully designed space for students to engage in learning prime factorization through structured play. "Darting Factors" was designed to engage students in exploring the unique role of 1, and the function of prime numbers in factorization, building on their existing understanding of factors. For metaphorizing, the dart context provides a familiar and engaging background, metaphorically representing prime factorization through a dartboard designed to generate any number by multiplying numbers hit by darts. For scaffolding, the tasks draw on students' prior knowledge of factors and are progressively structured to support the development of their understanding of prime factorization. For gamifying, dartboards with numbers and blank templates are created to foster conceptual growth regarding prime factorization through active engagement and deepening students' understanding of factors and prime numbers.

Accordingly, three stages of game-based tasks were designed. Stage 1 tasks (see Figure 5) require students to use all available darts to produce a target number, while prompting them to consider the role of 1 and to reflect on its unique property as the multiplicative identity. Stage 2 (see Figure 6) tasks challenge students to use as many darts as possible to reach a target number, this time on a dartboard that excludes 1, thereby encouraging strategic use of other factors, particularly prime numbers, to maximize dart usage. Stage 3 tasks (see Figure 7) invite students to design their own dartboards, enabling the generation of various numbers and identifying the power of each prime factor. Each stage supports specific conceptual goals. The Stage 1 tasks highlight the special role of 1 as the multiplicative identity, which does not alter the product. The Stage 2 tasks provide opportunities for students to perceive the unique function of prime numbers in maximizing dart usage and constructing efficient factorizations. The Stage 3 tasks foster an active connection between students' understanding of factors and prime numbers, guiding them toward formal concepts of standard prime factorization by visually representing the powers of primes on their custom dartboards within the gameplay environment. The evaluation of the game's impact on student learning is ongoing, with collected data currently being analyzed to assess changes in students' conceptual understanding.



Figure 5. Use all darts to produce a target number



Figure 6. Use as many darts as possible to reach the target number



Figure 7. Design a dartboard which can generate as many numbers as possible

Program #3: Just Learn Math (JLM)

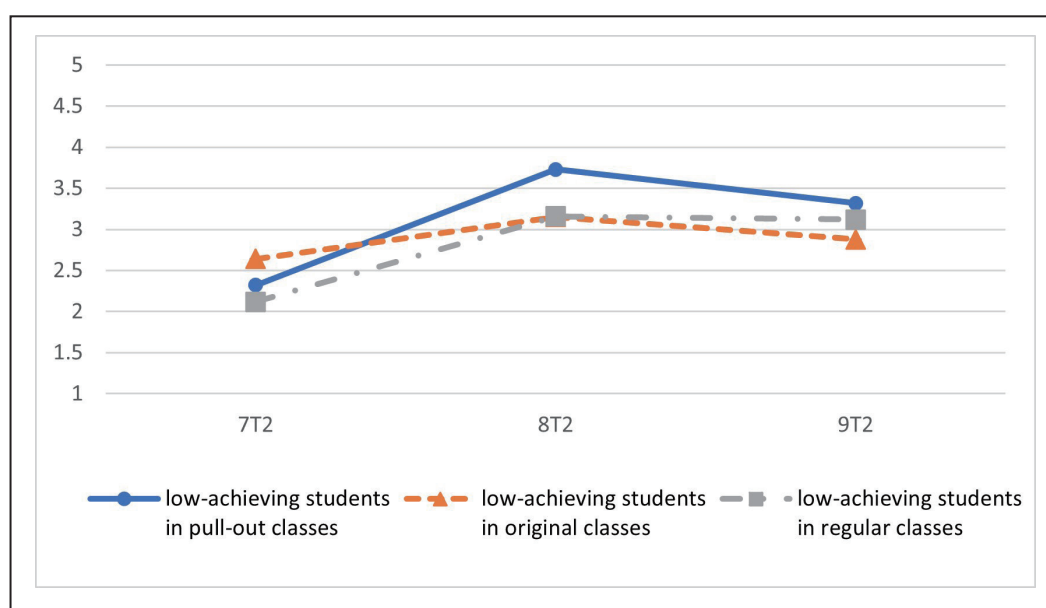
To address the wide variation in students' achievement and motivation, the Just-Learn-Math (JLM) program was launched in 2015. Its goal is to provide adaptive learning activities within the school setting, functioning as a formal education initiative that implements differentiated instruction by re-arranging learning environments. By organizing high- and low-achieving students into separate classrooms, the program aims to overcome the challenges teachers face in heterogeneous classrooms, where adjusting the pace and content of instruction to meet diverse learning needs can be particularly difficult. The JLM program implemented a pull-out strategy during mathematics classes to address the individual learning needs of low-achieving students who are motivated to continue learning mathematics. The number of students in the pull-out class is limited to 6 to 12, and the increase in teaching fees is covered by funding from the Ministry of Education's K-12 Education Administration.

Three key issues were tackled to bridge the wide variation in student achievement and motivation: how to group, how to teach, and how to assess. These design decisions are closely aligned with the overarching rationales guiding the program. First, the grouping criteria—which take into account both students' mathematics achievement and learning attitudes—are grounded in the rationale of *equity over equality*. Rather than applying a one-size-fits-all model, this approach ensures that students receive adaptive support and tailored challenges that meet their needs. It also reduces the instructional burden in pull-out classes while allowing the original class to benefit from smaller group sizes, thereby enhancing the learning environment for all. To mitigate the negative consequences of labeling, the program embodies the rationale of *prevention over remediation* by cultivating a positive and inclusive atmosphere from the outset. Key practices include engaging in open communication with parents, holding student interviews in a talent-show format to highlight strengths

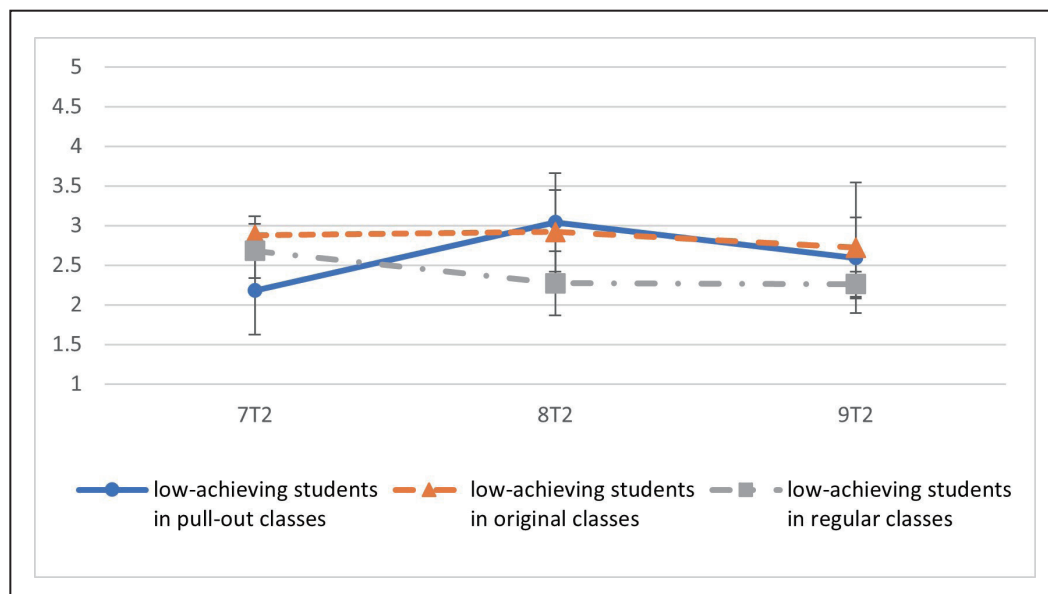
and attitudes, and naming the class “Enjoy-Learning Class” to reinforce a culture of motivation and growth. Finally, the JLM program reflects the value of *continuous improvement over superiority* by focusing on each student’s development, regardless of starting point. The grouping method fosters meaningful and appropriately challenging learning opportunities for both high and low-achieving students, not through comparison or ranking, but through sustained support for progress and persistence (Amit & Yang, 2015).

The instructional tasks were designed around two main types: guided exploration and guided reading comprehension. The former offered opportunities for students to construct mathematical understanding through the use of structured worksheets, encouraging active engagement and concept development. The latter supported learning through specially adapted textbooks designed for low-achieving students, aiming to build both content knowledge and reading proficiency. During worksheet implementation, teachers were expected to facilitate student discussions to deepen understanding through peer interaction. When using the adapted textbooks, teachers guided students in applying reading strategies such as predicting, clarifying, questioning, and summarizing (see Yang & Lin, 2012). To evaluate the effectiveness of the intervention, assessments were conducted to measure students’ attitudes toward mathematics, their use of learning strategies, and their mathematics performance.

Although low-achieving students in the JLM program scored significantly lower on the grade 6 basic mathematical competence test compared to students with similar rankings in regular classes, after 3 years of implementation, there was no significant difference in their performance on the grade 9 national mathematics comprehensive assessment compared to these students in regular classes. This finding indicates that low-achieving students can make substantial progress in achievement through participation in the JLM program. As shown in Figures 8 and 9, low-achieving students in the pull-out classes demonstrated the greatest improvement in attitude toward and strategy use for learning mathematics compared to low-achieving students in original and regular classes. This indicates that low-achieving students can make more significant progress in attitudes toward and strategy use for learning mathematics through participation in the JLM program.



Figures 8. Low-achieving students’ attitude across grades 7, 8 and 9 in one eastern county



Figures 9. Low-achieving students' strategy use across grades 7, 8 and 9 in one eastern county

The JLM program was scaled to approximately 60 schools (about 876 classes) in both 2018 and 2019, with 21 participating in the program during both years. To evaluate the effectiveness of the JLM program, one school was selected as a comparative school. A total of 3,344 eighth graders and 3,355 ninth graders participating in the JLM program were assessed, while 269 eighth graders and 382 ninth graders from the comparative school were assessed. Students participating in the program were allocated to the pull-out classes and the original classes. Figure 10 illustrates the mathematical ability growth trends in each kind of class from Grade 8 to Grade 9 after controlling for gender and school average ability variables. The initial ability of the eighth-grade original classes was comparable to that of the regular classes. As expected, the initial abilities of both the eighth- and ninth-grade pull-out classes were significantly lower than those of the original and regular classes. Overall, the original classes demonstrated the greatest progress, with an improvement of 1.6. The pull-out classes improved by 0.91, while the regular classes improved by 0.88, showing no significant difference in growth between the latter two kinds of classes. These findings indicate that the JLM program significantly enhances the mathematical ability of students in original classes, while enabling students in pull-out classes to maintain comparable growth with students in comparable classes, demonstrating its effectiveness in terms of supporting diverse learning needs.

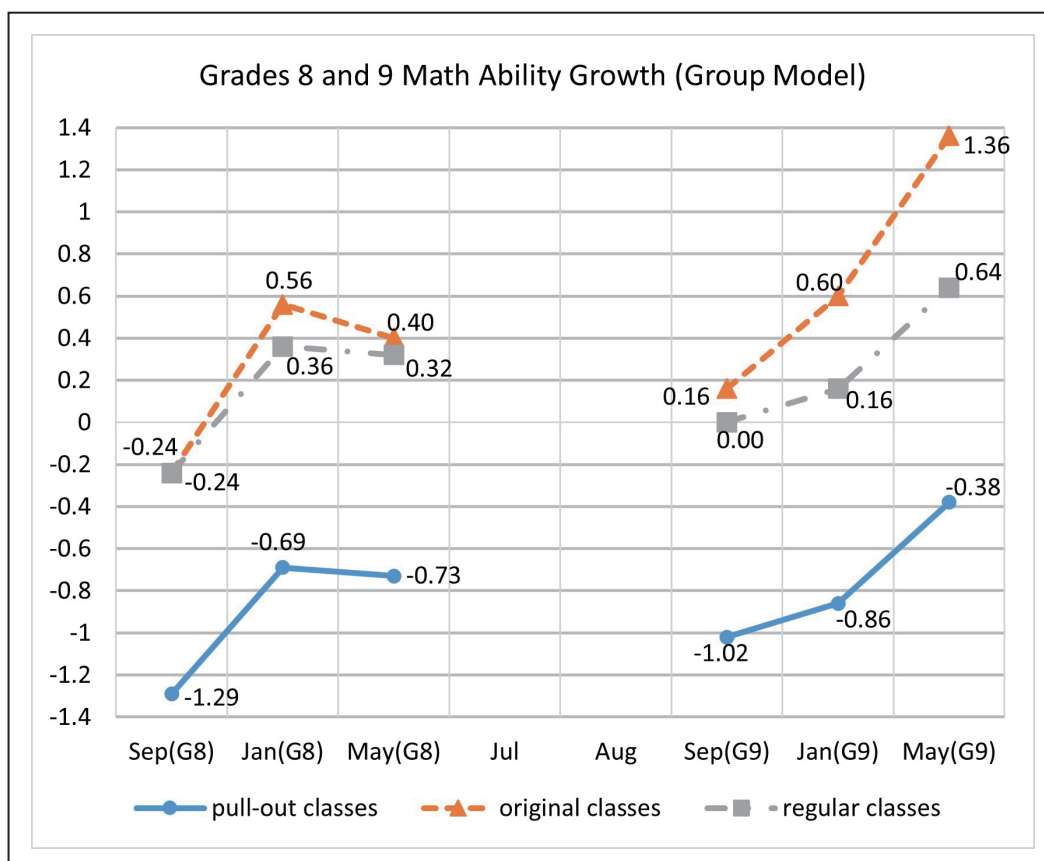


Figure 10. Growth trend of math ability across grades 8 and 9 (JLM)

Three key features characterized the design and implementation of the JLM program: (1) adaptive grouping complemented by adaptive learning, reflecting *equity over equality* by providing tailored support based on students' achievement and attitudes; (2) enhancing mathematics performance of low-achieving students through improved learning strategies—particularly reading—demonstrating a focus on *prevention over remediation*; and (3) multi-level learning pathways incorporating read/do, watch/write, and listen/discuss activities, which promote *continuous improvement over superiority* by offering diverse, structured opportunities for growth. Nevertheless, as of 2021, several challenges remained: (1) the need for one additional teacher for every six to seven regular classes resulted in the need for about 40 extra teachers in the JLM program; (2) limited changes in teaching practices were observed in original classrooms; and (3) low-achieving students who remained in original groups often received insufficient support.

Program #4: The Just Learn Math Program 2.0 (JLM 2.0, Extended from JLM)

Reflecting on the identified challenges led us to a guiding question: How can we enable more students to experience adaptive learning through teachers' differentiated instruction in their regular classes with the goal of reducing the economic burden on the government? This prompted the initiation of a new program aimed at developing teachers' competencies in digital differentiated instruction. Considering the continued growth of digital learning in the post-pandemic era, the "one tablet per student" policy, and the vision of cultivating lifelong learners in the new curriculum guideline implemented since 2019 (Ministry of Education, 2014), an experimental "adaptive grouping (inclusive model)" initiative was launched in the 2022 academic year. Experimental schools were invited

to participate in the JLM 2.0 program to support teachers in developing professional competencies in digital differentiated instruction in collaboration with professional project teams.

The same key issues, how to group, how to teach, and how to assess, were addressed to further narrow the wide variation in students' achievement and motivation. As illustrated in Figure 11, two primary student groups were recommended within the classroom. To support students in learning at their own pace, instructional videos were delivered via tablets, serving as an integral part of the teacher's broader instructional approach. Simultaneously, teachers were provided with a comprehensive collection of online resources for both instruction and assessment, enabling them to implement flexible, differentiated, and targeted teaching that responds to the diverse needs of learners.

To meet diverse learning needs, the program suggested three types of adaptive learning: foundational, applied, and advanced learning. Foundational learning supported students who needed more time by offering instructional videos and scaffolded worksheets for self-paced mastery of core concepts. Applied learning focused on real-world problem-solving by integrating individual reflection on video-based instruction, collaborative peer discussions, and teacher-led summaries to deepen conceptual understanding and facilitate the application of knowledge (Dai & Yang, 2024). Advanced learning targeted high-achieving students with enrichment tasks that extended beyond the standard curriculum, fostering independent exploration. Together, these approaches created a more equitable and responsive learning environment where all students could progress according to their needs and potential.

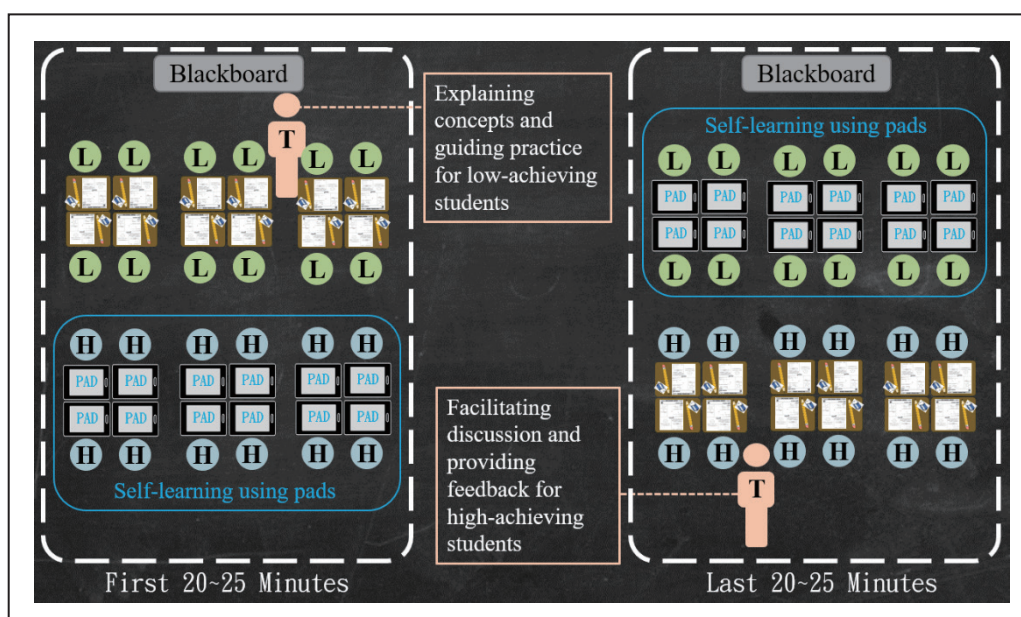


Figure 11. Classroom Configuration

As of 2024, 111 classes from nine schools had participated in the JLM 2.0 program. To evaluate the effectiveness of the program, 151 students of the experimental group from eight classes in two schools have been assessed since 2022. For comparison, 130 students of the comparative group from six classes in three schools have been assessed since 2022. Figure 12 illustrates the mathematical ability growth trends for each group assessed in Grades 7 and 8. Initially, the experimental group's ability was comparable to that of the comparison group. Over time, however, the experimental group showed greater progress, although it was not yet statistically significant. The findings

suggest that the JLM 2.0 program gradually enhanced mathematical ability, particularly in the experimental group, highlighting its potential for supporting diverse learning needs.

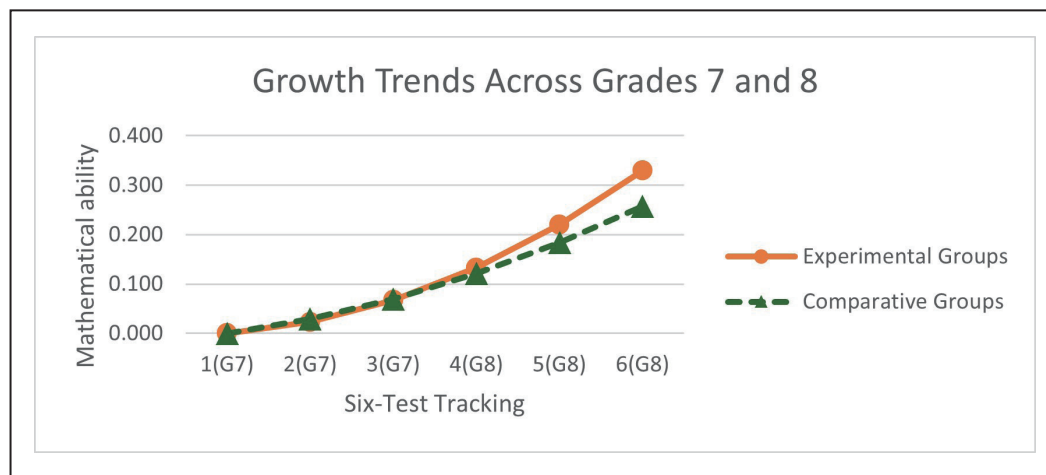


Figure 12. Growth trends in experimental and comparative groups across grades 7 and 8 (JLM 2.0)

SMILE as a Catalyst: Driving Forces for Bridging Two Achievement-Motivation Gaps

The driving forces underlying the JDM, QMS platform, JLM, and JLM 2.0 programs can be identified from each level of the bioecological systems (Bronfenbrenner, 2005) and reconceptualized through an enactivist perspective that views cognition as enacted, embodied, embedded, enacted, affective and extended (Ward & Stapleton, 2012). The driving forces are encapsulated in the acronym **SMILE**, which stands for *task Structure*, *adaptive Momentum*, *practice-research Interplay*, *responsible Leadership*, and *lifelong Engagement*. Each force corresponds to dynamic interactions situated within and across the microsystem, mesosystem, exosystem, macrosystem, and chronosystem levels of influence. Among them, Practice–Research Interplay functions as the pivotal force, enabling and sustaining the others through continuous, collaborative sense-making and adaptation.

Responsible Leadership: Distributed Agency Across the System

Responsible leadership in these programs reflects a shift from hierarchical control to distributed and relational agency (Harris & Spillane, 2013; Bennett, 2010). At the macrosystem level, leadership is enacted by a network of actors, including educators, researchers, school leaders, and policy advocates, who collaboratively guide the development of inclusive, adaptive, and sustainable mathematics education. Crucially, these efforts are anchored by government funding and strategic direction, which provide the necessary policy infrastructure, resource allocation, and legitimacy for innovation to become embedded within educational systems and scaled in a sustainable manner. Initiatives across the JDM, QMS, JLM, and JLM 2.0 programs underscore that responsible leadership is not confined to individuals, but rather emerges from systemic relationships, shared commitments, and the coordinated integration of human and non-human resources, such as MGAs, digital platforms, and tablet-based instructional tools. This form of responsible leadership not only shapes vision and direction but also ensures the structural and policy support required to initiate, sustain, and connect the other driving forces across the educational ecosystem.

Practice–Research Interplay: The Pivot for Systemic Co-Evolution

Functioning at the exosystem level, the practice–research interplay serves as the core pivot that activates and connects the other SMILE forces. It embodies the enactivist notion of participatory sense-making—a process where coordinated activity among individuals generates new understanding that transcends any single perspective (De Jaegher & Di Paolo, 2007). In these programs, this interplay facilitates reciprocal transformation between research-informed practice and practice-oriented research. For example, in the JDM program, teachers’ insights from real-world implementation inform research refinement, while theoretical insights guide instructional innovation. This ongoing loop makes it possible to bridge gaps between design and use—ensuring that tools such as task structures, instructional videos, and grouping strategies remain responsive to the lived realities of classroom contexts. As such, practice–research interplay serves as the foundational driver of innovation, profoundly shaping the co-evolution of responsible leadership, task structure, adaptive momentum, and lifelong engagement across all levels of the bioecological system.

Task Structure: Designing for Epistemic Action

In the JDM, QMS, JLM and JLM 2.0 programs, task design emphasizes the productive role of errors and cycles of inquiry to foster deeper understanding. Originally conceptualized as a framework of learning goals, processes, and motivation, task structure has evolved into a refined set of design principles and criteria (Yang et al., 2022b). From an enactivist standpoint, learning tasks are not mere content delivery mechanisms but epistemic actions—tools through which students construct meaning via interaction, experimentation, and reflection (Varela et al., 1991; Brown & Coles, 2012). At the microsystem level, well-designed tasks for closing the achievement-motivation gaps foster the co-construction of meaning between learners and tasks, effectively integrating affective engagement with cognitive development. These tasks are effective and meaningful when their design is continuously shaped by both research insights and classroom experiences.

Adaptive Momentum: Scaling Through Reflection and Co-Adaptation

Adaptive momentum captures the idea that sustained educational change arises from iterative reflection and mutual adaptation. Anchored in the enactivist concept of structural coupling, it recognizes that learners and their environments shape one another over time. For example, in the JDM programs, momentum was not externally imposed but emerged from teachers’ and students’ experiences—for example, when enthusiastic feedback on MGAs-for-camp prompted teachers to adapt them for classroom use. At the mesosystem level, adaptive momentum reflects how schools and communities co-develop practices in response to evolving needs and constraints. Crucially, this momentum depends on practice–research interplay to ensure that adaptations are informed, meaningful, and scalable.

Lifelong Engagement: Extending Learning Across Time and Systems

Finally, lifelong engagement reflects the chronosystem dimension of the bioecological model. It underscores a vision of mathematics education that extends beyond formal schooling, valuing learning as a lifelong, culturally situated pursuit. All four programs intentionally cultivate both cognitive and affective engagement, not only for students but also for teachers. MGAs and digital games are designed to spark curiosity and intrinsic motivation, while professional development opportunities enable teachers to engage in reflective practice and co-create meaningful tasks. Lifelong engagement also rests on the belief that learning strategies and mindsets—such as mathematical reasoning, communication, and reading strategies—are key to sustained learning. This engagement, however, is only made possible through task structure, adaptive momentum, a robust practice–research interplay, and responsible leadership.

Accordingly, the driving force behind each level of the bioecological systems is encapsulated as SMILE: *task Structure, adaptive Momentum, practice-research Interplay, responsible Leadership, and lifelong Engagement*. The SMILE framework promotes the reciprocal advancement of learning among students, teachers, educators, researchers and policymakers by creating, adapting and applying educational materials and technology. Herein, we highlight the interplay between practice and research, at the exo-system level, as a pivotal mechanism for activating and leveraging the broader bioecological system, and recognize mathematics-teacher-educator-researchers as key agents in initiating these programs, not as isolated actors, but as part of a dynamic and interdependent system.

Enacting SMILE across Systems: From Research-Informed Practice to Practice-Oriented Research

Rather than applying fixed principles to static learning environments, these programs co-evolve with the learners, teachers, researchers, policymakers, and communities they engage, and highlight the co-adaptive and participatory nature of meaning-making. Grounded in enactivist theory, SMILE embodies the belief that educational change is not transmitted top-down, but is enacted through reciprocal interactions between people and environments over time, supported by ongoing practice-research interplay. Then, teachers are invited to play multiple roles—not only as implementers, but also as co-learners and co-creators whose classroom experiences inform and refine pedagogical innovations. As they adapt these programs to meet their students' needs, they simultaneously deepen their own professional knowledge and instructional flexibility. Students, in turn, benefit from more responsive instruction grounded in research-informed strategies and implemented across both non-formal and formal educational settings. Students' engagement and progress offer critical feedback that shapes future iterations of program evolution to continuously improve feedback-informed resources and interventional strategies through practice-research interplay, like designing MGAs and MGA-based games through cooperation between teachers, educators and researchers, conducting professional development workshops using edited instructional videos, adapting textbooks and adjusting lesson structures supported by governments. By incorporating professional feedback, policy makers are able to make informed decisions that support both strategic planning and responsive adaptation to bridge the achievement-motivation gaps.

Mathematics teacher educator-researchers, situated at the intersection of research and practice with the connection to theory, engage in a dynamic process of theorizing practice and practicing theory (Lin et al., 2018), drawing insights from lived classroom experiences to develop frameworks that are both evidence-based and context-sensitive. The JDM and JLM programs were initiated, based on mathematics education educator-researchers' knowledge and experience in research to transform practice, as research-informed practice. To understand the mechanisms of learning and teaching and to evaluate their outcomes, practice-oriented research emerged. Then, the Quick-Witted Math Showdown Platform was proposed to expand the availability of digital MGA-based games, and its effectiveness was evaluated by students' self-reports, while JLM 2.0, adapted from JLM, was proposed to develop teachers' adaptive teaching competence and provide students with adaptive learning opportunities. Both programs evolved further from research-informed practice to practice-oriented research. Research-informed practice can delineate the gap between ideal research settings and real practice, while practice-oriented research can optimize the generalizability of research to practice.

Sustainable transformation emerges when practice shapes research (practice-oriented research), research informs practice (research-informed practice), and theories guide and evolve within research, practice and their interplay. It forms an iterative cycle in which students, teachers, educators, researchers, policymakers and educational materials co-evolve within the bioecological systems over time. Different from the perspective on theory as a mediator or a lens for informing or interpreting research, practice and their connections (e.g., Silver & Herbst, 2007), we count theory as a co-evolving entity for sustainable transformation, as theory shapes practice and research, and emerges from their interplay. The mathematics-educator-researchers' expertise acts as a catalyst for promoting learning and teaching quality to bridge the achievement-motivation gaps, combined with the iterative exchange among all stakeholders—students, teachers, researchers, and policymakers—by creating, adapting and applying educational materials, where theory is posited as a living, evolving entity embedded within educational ecosystems.

Conclusion

Two types of achievement–motivation gaps are commonly observed in East Asian countries which rank among the top performers in mathematics. Using Bronfenbrenner's bioecological systems as a guiding framework, this paper examines this phenomenon and analyzes four educational programs implemented in Taiwan aimed at addressing these gaps. Central to the development of these programs is the expertise of mathematics educator–researchers, who act as catalysts for iterative and co-evolutionary transformation. These programs are shaped and strengthened through dynamic interactions among five core drivers: task Structure, adaptive Momentum, practice-research Interplay, responsible Leadership, and lifelong Engagement, collectively conceptualized as the SMILE forces.

Underpinned by three key rationales: prevention over remediation, continuous improvement over superiority, and equity over equality, the SMILE forces offer a synergistic approach to enhancing the quality of mathematics education. In this context, synergistic refers to the mutually reinforcing integration of the five driving forces, each of which contributes more effectively in combination than as isolated interventions, particularly as they operate across and interact within the multiple layers of the bioecological system—from classroom-level interactions (microsystem and mesosystem) to broader institutional and policy environments (exosystem and macrosystem), and across time (chronosystem). This approach also envisions theory not as a static construct, but as a living, evolving entity that both guides and emerges from the interplay between research and practice. At its core, practice–research interplay functions as a pivotal connector—bridging human and non-human agencies, integrating systems within and beyond schools, and fostering a culture of continuous, responsive transformation.

A Möbius strip serves as a metaphor for this approach, as illustrated in Figure 13. Reflecting its connected, continuous surface and non-orientable structure, the SMILE forces merges “inside” and “outside” settings, dissolves the boundaries among bioecological levels, and reconceptualizes inconsistent orientations as part of a dynamic and co-evolving continuum. This metaphor powerfully captures the responsive interconnectedness and reciprocal influences that characterize an integrated and sustainable educational ecosystem. Such a perspective not only reconciles epistemic divides, but also synergizes human and non-human agencies across all levels of the ecosystem, laying the foundation for a more impactful and context-sensitive research paradigm in education.

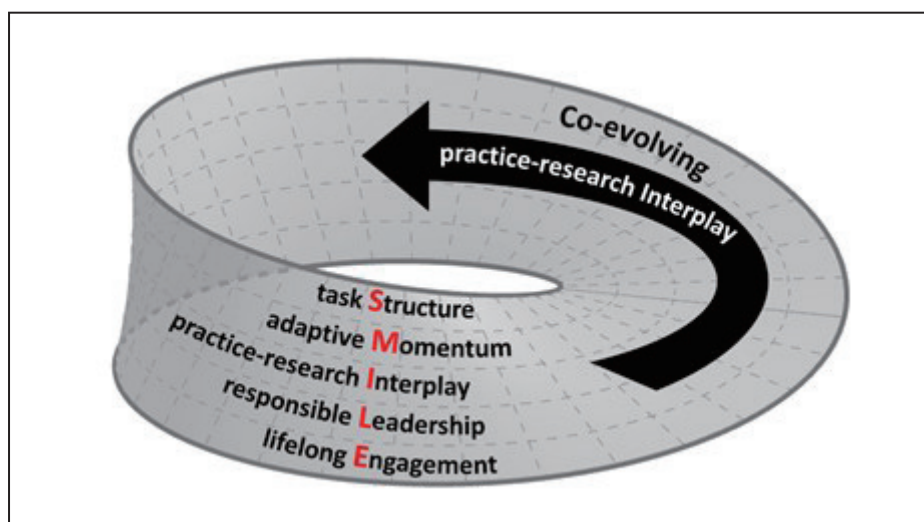


Figure 13. A Möbius strip as a Metaphor for a Synergistic Approach of the SMILE Forces

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Plenary Lecture 4

**A DESCRIPTIVE THEORY OF MATHEMATICS TEACHING:
WHAT CAN THE THEORY OF PRACTICAL RATIONALITY SAY ABOUT
THE IMPROVEMENT OF MATHEMATICS INSTRUCTION?**

Patricio G. Herbst



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



A DESCRIPTIVE THEORY OF MATHEMATICS TEACHING: WHAT CAN THE THEORY OF PRACTICAL RATIONALITY SAY ABOUT THE IMPROVEMENT OF MATHEMATICS INSTRUCTION?

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After providing a brief tour of the basic elements of the theory of practical rationality and how this theory supports and grows from descriptive research on mathematics teaching, I give an account of five key motivations behind the theory that underpin how it relates to practice. I then use those introductory remarks to elaborate on how the theory of practical rationality can be a resource for teachers doing research in their own practice, for policymakers seeking to design better reforms, for teacher educators seeking to empower teachers with tools to engage their stakeholders, and for all teachers looking to reflect on the predicaments of practice.

Keywords: improvement, norms, obligations, teaching, theory

Introduction

The relationship between research and practice is quite an important concern for mathematics education researchers. Beyond making contributions to theory and method, researchers want to produce usable and useful knowledge. To figure out what usable knowledge can be derived from a research program is, in itself, an intellectual effort. It is an effort that requires the researcher to be mindful of three things: first, the goals and needs of policymakers and practitioners; second, the limitations of scholarly knowledge in general and of specific research claims in particular; and third, the researcher's own position, as the transaction of the connections to practice of scholarly knowledge can be fraught with power imbalances. In writing this essay, I am also mindful of a fourth issue: I come from the United States, and I am writing for a conference of East Asian scholars. I cannot ignore the role that U.S. research in mathematics education often plays in the direction our international community takes when it conducts research—hence I need to second-guess how, as an American researcher talking about the connection between research and practice, I am not only talking about what I know, but also possibly suggesting how others should talk. And in doing that, I want to avoid coming across as an intellectual imperialist of sorts. Hence, my way to talk about the relationship between research and practice stays very close to the research I do and illustrates the connections to practice that I see in my work. Toward the end of the essay, I come back to the issue of research and practice in general as I address specifically the four points of caution named above.

For the last 25 years, I have been collaborating with Daniel Chazan in the development, through fundamental research, of a descriptive theory of mathematics teaching we call practical rationality (Herbst & Chazan, 2012). The main goal we address with the theory is supporting the generation of knowledge about mathematics teaching. Yet, as we make progress grounding theoretical ideas in data and finding that those ideas help explain teaching practice, we also wonder what we may have to say to practitioners about the improvement of practice. In what follows, I elaborate on how the insights acquired in the development of the theory of practical rationality of mathematics teaching might translate to policy and practice. The question that I posed myself when planning the presentation was, “What can practical rationality say about the improvement of mathematics instruction?”

and hence the subtitle. I am mindful of what kind of theory practical rationality is, so as a sort of warning that the connection to practice may not be so direct, the title raises some awareness—practical rationality is a descriptive theory of mathematics teaching, not a prescriptive theory: Its goal is to help researchers understand teaching practice, not to guide teachers into desirable practice. And yet, it can assist practitioners, including teacher researchers, policymakers, teacher educators, and all teachers as they do their work.

I have framed this discussion in the context of the research-to-practice relationship and noted the four reasons for caution as a way to organize the presentation. A traditional view of the relation between research and practice is described by the metaphor of a pipeline (Silver & Lunsford, 2017) according to which the results of research—canonically, the finding of a causal relationship between two variables where one of them can be controlled—have implications for what practitioners should do in the classroom with the variables that can be controlled. Research on practical rationality does not aim for this kind of prescriptive implication, but it does aim to be useful and usable. In offering this text, my expectation is that the reader will learn about practical rationality and its possible contributions to the worlds of policy and practice, and also that the reader will learn of a different way in which the relation of research to practice can be conceptualized.

Fundamental Research on Mathematics Teaching—What Is it?

Different approaches to research in mathematics education may call for different kinds of connections to practice. I have mentioned that I do fundamental, descriptive research on mathematics teaching, and the reader may wonder which words make a difference in that statement. I address that question first. I say research “on” mathematics teaching as opposed to research “in” mathematics education—that expression offers the preposition *on* instead of *in*, and the substitution of *teaching* for *education*. The preposition *on* ushers in a perspective whereas the substitution of *teaching* for *education* offers a focus.

I describe the research I do as research on mathematics teaching because there are other legitimate forms of research that deserve the description of being research *in* mathematics teaching—namely, the practices of lesson study that have been especially cultivated in East Asia, as well as the practices called action research, teacher research, and scholarship of teaching cultivated in the anglophone world. That is, teachers and university instructors themselves sometimes do research in the context of their practice; this research is often situated in practice and seeks to improve local practice. I come back to this valuable type of research toward the end of this presentation, when I describe how practical rationality could assist a practitioner doing research in their own practice. Research *on* mathematics teaching, however, seeks to understand the practice of mathematics teaching in general, often capitalizing on teachers’ accounts of their own practice in specific locations but aiming, fundamentally, to describe how teaching practice emerges in response to the conditions and constraints in which all teachers work.

I also describe the research I do as research on mathematics teaching and not as research on mathematics education or research on mathematics teaching and learning. This is because my focus is to understand the work that mathematics teachers do. To say the research is focused on mathematics teaching means that it seeks to understand all cases of mathematics teaching and not only the teaching that is instrumental for certain kinds of learning or the teaching that delivers a particular curriculum. Yet, it is neither possible nor desirable to extricate this work from mathematics learning and from the larger enterprise of mathematics education. Those elements are still present in the theorization, but only in the service of describing and explaining the practice of teaching mathematics. In

collecting and analyzing data, then, specific connections to the learning of particular mathematical ideas by students are essential.

Practical Rationality: A Brief Tour of Its Main Ideas

As I already mentioned, practical rationality is a descriptive theory of mathematics teaching: We explicitly designed it to capture the institutionalized position of mathematics teachers. Because of that, it is worth noting that we originally developed the theory in the U.S. to understand the work of high school mathematics teachers. Before saying more about the theory, and for the sake of my East Asian colleagues, the first section provides a bit of context about high school mathematics teaching in the U.S.

The Context of High School Mathematics Teaching in the United States

I provide the following broad overview with the caveat that there is a lot of variability in the country. High school mathematics teachers in the U.S. are usually expected to have a university degree equivalent to a mathematics major. They are organized in mathematics departments and teach courses of mathematics study to different groups of around 30 students every day, usually for periods close to an hour. A high school mathematics teacher will typically teach three different subjects to five different groups of students each day (thus, they may teach some material more than once a day but must attend to the learning needs of five groups of students). Teachers commonly use a published textbook from which they draw lessons as well as homework problems for students. Teachers usually work alone in their classrooms but occasionally have the assistance of a special education teacher to help students with special needs. Courses are usually yearlong and focus on areas of mathematics (e.g., geometry, precalculus). Teachers give students a final grade which is recorded on the students' transcript.

Practical Rationality: Its Goals

When we developed the theory of practical rationality, we sought to contribute to the scientific study of mathematics teaching. This means that we sought to create concepts that help name some of the sources of variability in practice. Those concepts describe attributes of the position of teacher; of the engagements between teachers, other agents (e.g., students, administrators), and content; and of the regulations for those engagements. Practical rationality uses those concepts to further describe the connections among aspects of practice, explaining those connections in terms of relationships that could include causal ones. And it aspires to predict aspects of the work of mathematics teachers in the classes they teach, particularly decisions that have to do with the management of students' learning of content. The development of measures supports the work of describing, explaining, and predicting. Measures attempt to capture the variability with which individual teachers relate to these various aspects of their work and support the modeling of relationships among concepts using the statistical tools of social research.

Any work of this nature is perforce reductive of the complexity of actual practice and so its assertions are always tentative. The assertions of practical rationality are not primarily about real practitioners or real classrooms, but about concepts that identify some of the characteristics of real practitioners and real classrooms. The theory aims to understand how various complexities of the work of teaching are related—not to diagnose or judge individual teachers.

Practical rationality posits that the work of teaching is an outcome of the adaptation of agentic individuals to the conditions and constraints of their work. Thus, among the concepts that it posits to describe the work of teaching, some concepts model the sources of individual agency (e.g., knowl-

edge, beliefs) and some concepts model the sociotechnical conditions and constraints of work (e.g., obligations, instructional exchanges, norms).

Sources of Teachers' Individual Agency

To model the sources of individual teachers' agency, we rely on research in our field that describes teachers' experience teaching in various contexts (e.g., courses, levels of schooling, schools), their professional preparation (e.g., the mathematics courses they took, the sort of teacher preparation they received), their beliefs (about mathematics, about teaching, about learning), and their mathematical knowledge for teaching. These concepts can be operationalized in terms of several variables. For some of those concepts, self-report data (e.g., how many years someone reports having taught a course of studies such as high school geometry) may be sufficient. For other concepts that name latent traits (e.g., mathematical knowledge for teaching, beliefs) we rely on theories and measures developed in the field of mathematics education research or in educational psychology. We have relied on Ball et al.'s (2008) conceptualization of mathematical knowledge for teaching and adopted or developed measures to estimate the amount of mathematical knowledge for teaching of specific courses of study, particularly algebra (Ko et al., 2024a) and geometry (see Herbst & Kosko, 2014; Ko & Herbst, 2020a). Likewise, we have relied on measures of teacher beliefs developed by others (e.g., Stipek et al., 2001) when inspecting the role of teacher beliefs in decision making (Ko & Herbst, 2020b).

Sociotechnical Conditions: Professional Obligations

The original contribution of practical rationality is in modeling the conditions in which teachers work. These include what we refer to as sociotechnical resources, including regulations that concern how teachers interact with mathematical and other knowledge and with other agents in educational institutions. We assume that individual teachers take on a professional position when they accept a job teaching mathematics at a school. In this position, they are accountable to four stakeholders: Knowledge, the Client, Society, and Organization (Chazan et al, 2016).

As the name for a stakeholder, *Knowledge* refers especially to mathematics as the discipline and practice of knowledge creation from which the content of studies derives its legitimacy: We posit that mathematics teachers are obligated to the discipline of mathematics as the ultimate source for what they teach and as a source of justification for their choices of content. Thus, the discipline consists not only of the knowledge edifices created by mathematicians but also of the practices that mathematicians engage in to create or use their knowledge and the practices that other users of mathematics (e.g., engineers) need it for. Teachers can use the discipline as a source of justification to shape their instruction, and they are accountable to the discipline when they represent mathematical knowledge and practice in class. This *disciplinary obligation* manifests itself, for example, when students ask for a definition of a term or for the reason something is true. We posit that, because of this obligation, a mathematics teacher is expected to gravitate to definitions or justifications that are mathematically appropriate.

The second stakeholder, *the Client*, refers to individual students as the direct beneficiaries of the work teachers do. We posit that mathematics teachers are obligated to individual students as those who can benefit or suffer from the work of mathematics teaching. We see this obligation to individuals not only in cognitive terms, where the impact on students may involve understanding mathematical ideas or not, but also along other individual dimensions according to which students may or may not benefit. What those other dimensions are can vary culturally as well as historically. In the U.S. during the 21st century, these dimensions include physical health (e.g., obligation not to

risk students' health or safety), emotional well-being, and motivation (e.g., obligation to support students' motivation to study in school and their sense of selves). The dimensions that a given culture acknowledges individual children and youth to have needs to be considered when modeling what the obligation entails for a mathematics teacher in that society. This *individual obligation* is manifest in the expectation for teachers to attend to individual needs and wants when interacting with individual students (e.g., when giving feedback or personalizing assignments) as well as when planning activities in which students will be involved (e.g., when deciding on the various sources of difficulty of those activities).

The stakeholder *Society* refers to the needs, norms, and values of the nations and communities that support schools, send their children and youth to schools, and benefit or suffer from the education that children and youth have received in school. We posit that the teacher is obligated to society in several regards, though the specific components of that obligation may depend on the society itself—particularly on the role that schooling plays in each society (e.g., whether education is compulsory, free, etc.). In the U.S. during the 21st century, students of mathematics are expected to learn skills that they will use in the world of work, including mathematical literacy for the use of equipment, elements of work ethic such as honesty and accountability, social skills to interact with people at work and in society at large, and so on. As an adult in the classroom, the mathematics teacher is expected to be a steward of society, demonstrating and enforcing social values (e.g., fairness) and disowning social ills (e.g., racial prejudices). These values might include, in some cases, community values (e.g., charity) that a specific school (e.g., a religious school) may aim to develop in students but may not be shared to the same degree with the society at large. Practical rationality names the obligation to society the *interpersonal obligation*, because its presence in the classroom is often manifest in how the teacher handles microsocial issues such as sharing resources, sharing their own time between students, and managing the sharing of public discourse.

Finally, *Organization* refers to the expectations that a position in an organization establishes for the individual teacher, including subordination to institutional supervisors (e.g., department chair, principal), attention to effectiveness goals of the organization (e.g., helping increase test scores), attention to policies and practices of the institution (e.g., bell schedules, exam schedules, curriculum choices), contribution to schoolwide climate and functions (e.g., respecting and collaborating with colleagues), and contribution to the stability of the school in the larger society (e.g., collaborating with law enforcement). As in the case of the interpersonal obligation, some institutions may expect teachers to do things that other institutions do not (e.g., follow a specific pacing guide for the curriculum, use a particular pedagogy for group work). Practical rationality names this obligation *institutional* because it ties the individual teacher to the various institutions within which mathematics instruction is situated.

These four obligations (disciplinary, individual, interpersonal, and institutional) describe regulations associated with the position of someone who is hired to be a mathematics teacher in a school. Among the things the school does is to carve out room for a key engagement: Instruction. The school does so by organizing courses of studies, staffing those courses with teachers to teach them, and assigning students to take those courses. Because those courses are part of a multiyear experience students have in the institution, the instruction a teacher is engaged in is shaped by those institutional characteristics—for example, the sequence of courses students take is expected to affect the teacher's choice of what content to teach in their own course. We posit that the way a teacher engages in instruction develops as they use their individual resources to respond to the obligations. But it is relevant to define instruction as an engagement independent of the obligations that shape it, because some things are common to instruction regardless of the institution in which it happens.

Instruction and Instructional Exchanges

Instruction is an engagement in which a mathematics teacher relates to and interacts with students and content. Cohen et al. (2003) defined instruction as a system of interactions among three components—teacher, students, and content—in environments. In practical rationality, we build on that definition by attending, in particular, to how instruction happens over time and involves actors who have dissimilar relationships with the content over time. Whereas the teacher knows the content from the beginning of the interaction, the student initially does not but expects to come to know it over time. Instruction engages the student with the content so that, as a result of such engagement, the student has the opportunity to come to know the content. The teacher designs and manages that engagement.

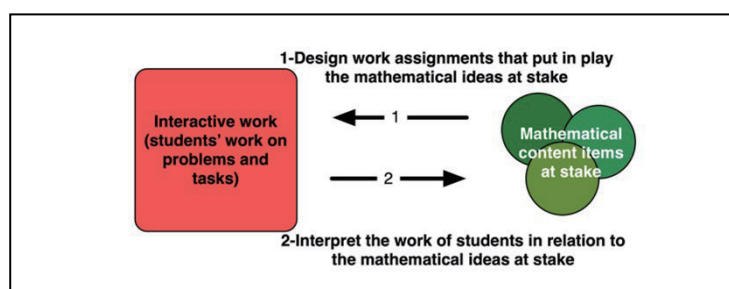


Figure 1. Instructional exchanges

On the basis of these considerations, a key aspect of the teacher’s work in instruction is to manage instructional exchanges. These are exchanges of content-related meanings and are operated by the teacher, who is able to see the content in two different ways (see Figure 1). The teacher can see the content, first, as items of knowledge to be learned—for example, as Standards or curricular goals, usually with conventional names such as “angles formed by systems of parallel lines intersected by transversals.” But as the designer and manager of instruction, the teacher can also see the content in the form of the mathematical work that students may spend time engaged in. Note that the first element of the exchange is timeless, whereas the second is necessarily timebound.

A trivial example of such a timebound work assignment is listening to a lecture (or viewing a video) that exposes the content—in which, say, the various angles formed by systems of parallel lines intersected by a transversal are defined and exemplified and their properties are written out. By attending to that exposition over time, the student might have the chance to come to know the content. The work of listening, for example, requires the student to discern which words are technical and which ones are not, how words relate taxonomically, and more. The exposition fleshes out the curricular goal over time and introduces the role of the student, who has to engage with it in order to realize the learning opportunity.

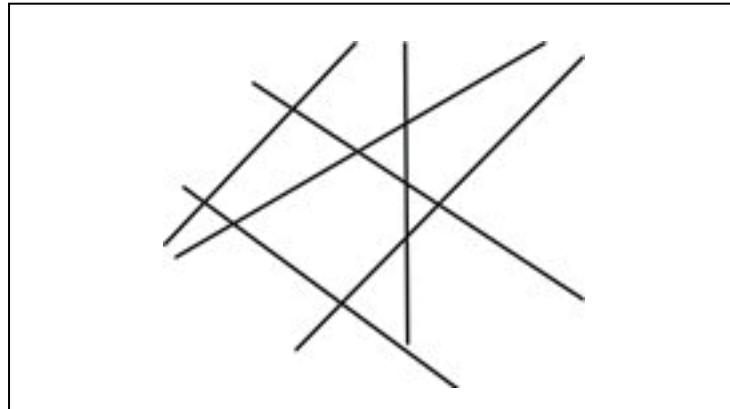


Figure 2. Diagram provided with the question “How many angles would you have to measure in order to know the measures of all of the angles between these lines?” (see Chen & Herbst, 2013)

Of course, listening to expositions is not the only way in which students engage with content. Frequently, students work on problems or exercises chosen because they provide opportunity to learn or use the content at stake: The teacher could assign a problem to students (such as the one offered in Figure 2), in which, as a result of their engagement, the students could show that they know the content (or how far they have come to know the content).

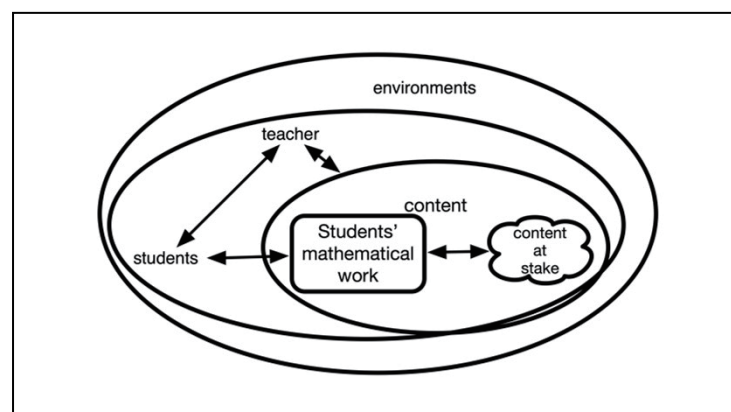


Figure 3. The instructional triangle and instructional exchanges

More generally, an instructional exchange consists of the potential equivalence between a work assignment completed over time by students and the item of content purportedly at stake in that work assignment. We say that the content is at stake because, to effect the exchange, the teacher will need to ascertain that the students’ completion of the work assignment has satisfactorily made use of the content on whose behalf the work had been chosen. This may or may not happen: For the work suggested in Figure 2, if too many students measured all angles in the figure using a protractor, the teacher could not validly say that students know the properties of the angles formed by transversals intersecting parallel lines. Thus, instructional exchanges are, in practical rationality, a key element in the modeling of the content of instruction. Figure 3 represents the instructional triangle with the instructional exchange in place of the content.

Instructional Norms

A key aspect of the work of the teacher in instruction is the management of instructional exchanges between the content at stake and students' work with content. This management could be a part of any kind of instruction, in schools or outside of them. But in schools, because the timeline of instruction is usually longer than a single exchange (e.g., students stay for several years, courses run for several months, the teaching of a topic takes several days), norms develop for how students and teachers engage with content and each other. These norms come with students from prior courses of study and are further shaped in interaction with the teacher for work with new content in a given course of studies.

Practical rationality uses Brousseau's (1997) concept of the *didactical contract* to refer to the set of norms that regulate instructional exchanges in mathematics. There is a tacit agreement between teacher and students regarding content, whereby the teacher is responsible to teach the content to students and students are expected to learn the content from the teacher. We use the word *norm* descriptively to mean habitual ways of doing things that teachers and students perceive as appropriate. For example, teachers and students usually see it as appropriate for mathematics teachers to assign work to students; for students to pick what work they want to do would probably not be seen as equally appropriate in mathematics classes, though this happens in other academic areas (e.g., in reading, students sometimes choose what they read).

These general norms of instruction we call (global) *contractual* norms. They tend to be stable over long periods (e.g., a year), but sometimes they change: For example, one such norm that describes instruction in many classrooms is that when teachers assign problems to students, these problems are expected to be solvable with the knowledge students purportedly already have. However, in some classrooms, students may also expect that the knowledge they have may not be enough to solve some problems and they may expect that they will have to figure out something new. This difference in didactical contracts may occur between different classes but it might also evolve over time within the same class—say, to make the second kind of problem something students can expect. This potential difference between contracts is described in practical rationality in terms of differences in norms.

Thus, different didactical contracts can be identified. These are different general ways in which students and teacher interact around content. One contract that is common in the U.S. is what Herbst et al. (2023) called *teaching through demonstrating problem types*, whereas another, *teaching through problems*, is less common but of interest to investigate. Those two contracts can be described as systems of norms that contain different norms regarding what sorts of problems teachers may assign students to work on. Norms like these are global or contractual in the sense that they describe what is expected across many different instructional exchanges. But there are also norms that describe what is expected in particular types of instructional exchanges.

Instructional Situations and Their Norms

As we learned from Brousseau (1997), the engagement of students in specific mathematical work often requires negotiating a local didactical contract—that is, negotiating the conditions for students to take responsibility for a particular problem. These conditions may include what sort of resources are given, what sort of product students are expected to produce, how long it might take to do it, what operations are appropriate for students to take, and so forth. Again, because instruction happens over time, the most important resources that teachers and students rely on for negotiating new local contracts are the prior local contracts students have experienced. New work assignments can

evoke past work assignments and, hence, also invoke the norms of a previous local contract. New work assignments may involve explicit revisions of some norms or may require a revision down the line, as the work is being done and the students or the teacher realize that the original expectations cannot hold. The concept of *instructional situation* names local didactical contracts that are recurrently evoked to regulate specific instructional exchanges.

By *instructional situation* we refer to a local didactical contract whose norms can be invoked by the way in which the work assignment is described, thus waiving the need to negotiate specific norms for students' engagement. *Calculation with formulas* is an example of an instructional situation: Students may be given a diagram or representation of a plane figure or solid (e.g., a circle or a pyramid) with some of its numerical dimensions (e.g., the length of the radius of a circle) and asked to determine a measure of it (e.g., its area, perimeter, surface area, or volume). Students are expected to retrieve a formula, identify variables in the formula with some of the dimensions given, and then calculate the formula using those dimensions. Many concepts can be learned using this situation and the situation might also be invoked to solve novel problems (e.g., given a triangle, how can you find (a) another triangle that shares a side and has the same area? (b) all the triangles that share the same side and have the same area?), though some negotiation of what students are expected to do may be needed (e.g., do they need to actually calculate an area to determine that two triangles have the same area?). Each of the norms of an instructional situation states what the teacher or the students are expected to do. For example, the teacher is usually expected to provide enough information for students to be able to feed the formula. A deviation from a norm may require some negotiation of the local contract, but many other things can be done without further discussion by recourse to the instructional situation being evoked. In practical rationality, we often refer to instructional situations as *frames* for selected instructional exchanges.

The norms of instructional situations (and contracts) are elements of tacit collective knowledge in teaching (Collins, 2010); they become actual statements when researchers model those situations and contracts. These statements of norms are one particular way in which practical rationality achieves being a subject-specific theory of teaching mathematics. The norms that describe what teachers are supposed to do in different instructional situations are mathematical in nature, as exemplified with the case of the norms of calculation with formulas. The relationship between instructional situations and mathematical content is a many-to-many relationship: To teach a concept or a property, a few situations may be needed; at the same time, the teaching of many different topics may be supported by the same situation. And situations may morph into new situations. For example, calculating with formulas is a natural predecessor to what Hsu and Silver (2014) called *geometric calculation in number*, in which students are given diagrams of known figures with some numerical dimensions for sides or angles and asked to determine the numerical dimensions of other sides or angles. The students are expected to retrieve properties they know of those figures and use them along with the known dimensions to infer or calculate the unknown dimensions.

Ball et al. (2008) introduced the expression “mathematical tasks of teaching” to name the actions that a mathematics teacher routinely does with mathematical content, such as analyzing student errors, providing examples of concepts, or choosing problems for students. In practical rationality, these mathematical tasks of teaching are building blocks to spell out the norms of didactical contracts and instructional situations. As noted in the definition of instructional exchanges, two key tasks of teaching are (1) creating problems for students and (2) understanding students' work. The management of instructional exchanges requires the teacher to choose or design work engagements for students in service of items of the content at stake. This management also requires the teacher to interpret students' work in service of the content at stake.

What Research Does the Theory Support with These Concepts?

The canonical role of a descriptive theory of teaching is to support research that describes and explains the work of teaching. One kind of research seeks to discover or provide evidence for general relationships among the constructs of the theory. For example: How do norms of instructional situations relate to norms of the didactical contract? In what conditions might these norms support each other or contradict each other? What are the challenges that these relationships pose for a teacher? A second kind of research seeks to characterize specific contexts such as contracts or situations. For example: If doing proofs is an instructional situation in high school geometry, what are its norms and what are the management costs that a teacher incurs if they allow departures from these norms? A third kind of research seeks to predict teachers' decisions and actions in selected contexts. For example: How are teachers of a course of study (e.g., algebra I) likely to carry out particular tasks of teaching (e.g., assigning a problem to students, responding to students' work) in given instructional situations (e.g., when solving equations)? If other elements of the context might prime considerations of one or another professional obligation, how much does the teacher's recognition of those obligations matter in explaining the actions taken or the decisions made? Are such predictions of what a teacher would do the sole function of the individual assets of an individual teacher? Or how does recognition of the norms of an instructional situation or of the professional obligations matter in the decisions that teachers make? That is, the research done with the theory seeks to further develop knowledge about the theory: Once concepts are proposed to describe the work of teaching, what else can we find out about teaching that can be described using these concepts? The results of this research are theoretical in the sense that they help the theory gain texture.

For the sake of pursuing such fundamental research, even more basic research has been needed. Specifically, we wanted to turn the concepts that describe the conditions in which teachers teach mathematics into measurable constructs that can be used in empirical research. This included the development of instruments that measure teachers' recognition of norms and obligations. Because norms are conceived as tacit elements of collective knowledge (Collins, 2010), the items that measure teachers' recognition of norms solicit teachers' responses in the context of scenarios that depict actions in classrooms where norms are either complied with or breached. We gathered evidence in favor of the hypothesis that a proposed norm describes what is expected in an instructional situation from teachers' descriptions of scenarios where the norm had been breached (see Dimmel & Herbst, 2018; Herbst & Chazan, 2015). Similarly, the measurement of recognition of professional obligations includes scenario-based assessments in which teachers are asked to rate the extent to which they approve of an action by a teacher who departs from a contractual norm to honor a particular obligation (see Ko et al., 2024b). Finally, to investigate how these constructs help account for decision making, we developed a set of items that prompt teachers to make decisions in the context of carefully designed scenarios (see Herbst et al., 2016; Ko & Herbst, 2020b).

The specific references provide more concrete details of the research results, but as an example, I cite what Ko et al. (2024b) found using the PROSE instrument, which measures teachers' recognition of each of the obligations. We administered this scenario-based assessment to high school mathematics teachers and to instructors of lower division college mathematics courses, finding that: (1) High school teachers have a higher recognition of the individual and interpersonal obligation than college instructors; (2) college instructors have a higher recognition of the institutional obligation than high school teachers, though there is also more variability in that recognition among college instructors; and (3) we found no differences in the level of recognition of the disciplinary obligation across groups.

Having done a tour of the theory of practical rationality, I want to take a step back and consider what could be described as the ideological basis for the theory. This will pave the way for the question in the title of the talk—what does it have to contribute to the improvement of instruction?

Practical Rationality: Its Motivations

Practical rationality fulfills some motivations that can be described in terms of intellectual resistance both to the mainstream of mathematics education research and to the usual ways in which research relates to practice, particularly its participation in mathematics education reform. The following five sections express critiques to these mainstream positions and describe how practical rationality resists them.

Practical Rationality Resists a Voluntaristic Conception of Mathematics Teaching

Practical rationality questions the extent to which mathematics education research and mathematics education policymaking rely on teachers' deliberate will as an explanation of actions and a motor of change. Mathematics education research usually frames students' learning as a natural occurrence, amenable to descriptions that rely on individual development and adaptation to the environment. But the work of teachers is often framed as a projection of the teacher's will, or as Goffman (1974) would call it, a guided doing. Some researchers of mathematics teaching have conceived of this work as an enactment of the goals, beliefs, and knowledge of the individual teacher (e.g., Schoenfeld, 2023). According to that conception, the qualities of someone's teaching reflect their knowledge and beliefs and, therefore, the enterprise of improving instruction consists of replacing teachers' beliefs and providing them with the knowledge so that they will want to implement reforms (Battista, 1994). This way of reducing social activity to the agency of the actor is what I refer to as voluntaristic.

Practical rationality avoids voluntarism by positing, instead, that, although teachers' individual assets matter in how they respond to the conditions and constraints of their work, their work is structured in ways such that what they end up doing requires an adaptation to their context rather than an execution of their will (Skott, 2009). Practical rationality posits that just as there are systematic phenomena in students' learning of mathematics (e.g., predictable, conceptual errors associated to particular concepts, such as subtraction), there are phenomena of mathematics teaching that cannot be reduced to the will or the individual characteristics of teachers.

Practical Rationality Resists a Generic Conception of Mathematics Teaching

A quality of good research in mathematics education is its relatedness to mathematics, not just as context for generic pedagogical phenomena but as a source of the complexities to be understood (Kilpatrick, 1995). Mathematics education research on students' learning and on curriculum has tended to be mathematically specific. Research on mathematics teaching, however, even when done by mathematics education researchers, has had some difficulty being mathematically specific. For example, accounts of teachers' noticing, discursive moves, or management of tasks have often employed languages of description that assume genericness to the tasks that teachers do.

In practical rationality, we recognize that there is economy in genericness and that some generic language (e.g., the notion of activity type; Lemke, 1990) is useful. Indeed, practical rationality does not aim to describe everything a mathematics teacher does, but is more concerned with what the teacher does that plays a role in creating mathematics learning opportunities (Hiebert et al., 2025). Along those lines, practical rationality questions the usefulness of generic teaching moves such as those to manage classroom discussions of student contributions, because mathematical sensibilities

suggest that any generic catalogue of teaching moves (e.g., O'Connor & Michaels, 1996) used to handle the content of instruction might contain important ambiguities. For example, a teacher asking “why is that?” of a student who writes the statement, “the angle bisectors of a parallelogram make a rectangle” is doing something substantively different from a teacher asking the same question of a student who adds a quantity to both sides of an equation. Although the discursive move in both cases may be one that seeks justification, the kind of justification sought is unlikely the same: As Chazan and Sandow (2011) showed, the first is likely to seek a justification that a statement is true, while the second a justification that the addition is strategic or convenient.

Practical rationality aims to model the mathematical sensibility with which a teacher engages in tasks of teaching mathematics—to refine a language of description that can distinguish between linguistically similar occurrences according to their uses in the sociotechnical contexts of mathematics teaching that vary especially according to mathematical features. These variations are not necessarily topical, but contexts are likely to be determined by the type of mathematical work a teacher has to manage. Practical rationality characterizes the types of mathematical work in which generically named tasks of teaching (such as creating problems for students or interpreting student work) require further specificity in order to describe more and less sensible enactments by a practitioner. As noted above, instructional situations and their norms provide examples of the granularity of the desired language of description.

Practical Rationality Resists a Conception of Teaching as Simply Instrumental for Learning or Curriculum

Mathematics education researchers became interested in teaching some time after becoming interested in curriculum (e.g., what are the mathematical meanings made available to be learned?) or in learning (e.g., how do students develop understandings of mathematics?; see Kilpatrick, 1992). In those latter two families of research traditions in mathematics education (which might be called didactical and psychological), as well as in much of the so-called production function, econometric studies of instruction (e.g., Crawford et al., 2012), scholars have found it valuable to incorporate research questions about the role of the teacher. Chazan et al. (2016) called those studies of teaching *instrumental* because they ask questions about the role the teacher plays in the classroom construction of mathematical ideas, the development of students’ understanding, or the increase in students’ achievement. Those kinds of studies are important, but practical rationality aims to do something different—to support a fundamental study of mathematics teaching.

Practical rationality aims to account for the work of teaching as an outcome of the engagement of teachers in work that exists in contexts characterized by conditions and constraints, among which are the multiple expectations of instrumentality noted above (e.g., the expectation that they will represent mathematics, help children learn, raise achievement), as well as others. Practical rationality does not assume that a moral hierarchy exists among those expectations, that those expectations always cohere, or that it is possible for a teacher to meet them all optimally. Instead, it acknowledges that mathematics teachers work to meet all of those expectations and aims to model how they do so. Practical rationality identifies the stakeholders of the work of teaching and how they obligate a teacher to the point that the capacity to meet the ends of any of those stakeholders depends on how practitioners accommodate and reconcile those different expectations to reach some equilibrium.

Practical Rationality Resists Methodological Fundamentalism

Mathematics education research shows an uneasy relationship between theory development and theory testing that relies on too sharp a divide between those who use quantitative and qualitative methods. One side levies the critique that all one can learn from inquiring into participants’ mean-

ings are idiosyncrasies; the other side suggests that measurement and statistics cannot reveal anything mathematically meaningful. Both positions seem to unduly deepen the divide between etic and emic perspectives. Fortunately, our field as a whole is methodologically pluralistic and a range of research methods are actually available to be used if researchers are willing to navigate across this range of methods in the service of a research agenda.

We believe that qualitative work is essential to discover important phenomena in how teachers experience the tensions and contradictions of practice. Indeed, part of the motivation for our development of the theory came from Dan Chazan's realization that, as a high school teacher, he saw himself having different mathematical expectations for mathematical work with students studying different topics, even when this happened in the same institution and sometimes with the same group of students in different years (Chazan, 2000; see also Eisenmann & Even, 2011). Yet, understanding the difference between the details of what those phenomena involve in general and the idiosyncrasies of particular people requires being willing to define measures and examine how those measures vary at scale. The development of practical rationality was also motivated by the challenge that Chazan and I read in the report *Scientific Research in Education* (Shavelson & Towne, 2002), which appeared to suggest that mathematics education research on teaching that aspired to be scientific would necessarily be applied research, that it had to be circumscribed to evaluations of the efficacy of mathematics education interventions (Herbst & Chazan, 2023). We questioned that suggestion by positing, instead, that mathematics education research on teaching should be able to study fundamental questions about practice from a perspective that did not drop mathematical specificity but examined questions with all the tools of social research. We thought that our whole field should accept the challenge to measure and quantify and explore basic relationships among constructs using the full range of methods in social research. Thus, practical rationality not only uses rich data from case studies to theorize abstract structures (e.g., the notion of instructional norm to name each of the expectations that teachers appear to work under), but it also develops operational constructs (e.g., recognition of a norm) and techniques (e.g., scenario-based assessments) that can be used to study those structures statistically and add evidence for or refute theoretical predictions.

Practical rationality acknowledges that knowing through research necessarily involves reductions of complexity. The settings in which and instruments with which we study mathematics teaching are among those reductions, which Kennedy (1999) characterized as approximations of the real circumstances in which practice occurs. In practical rationality, such reduction is apparent in our use of scenario-based assessments and simulations to elicit and measure teaching and in the use of statistics to make claims about practice (Herbst et al., 2022). At the same time, those reductions afford opportunities to address some of the challenges that researchers have identified in studying the relationship between knowledge and practice (Charalambous, 2020). Simulated engagements (such as scenario-based assessments and teaching simulations) are, for us, comparable to the task-based interviews used by researchers on mathematical cognition—occasions for the controlled study of practice that, because of the controls that can be exercised (e.g., one can simulate a specific situation and the breach of a specific norm just as much as a cognition researcher can choose tasks of a particular kind), permit us to reproduce practice at scale.

One might still wonder why that research is needed. Beyond responding to the challenge to demonstrate that theory-building research on teaching can be mathematically specific and use quantitative methods, this research is needed to support a bidirectional relationship between theory and practice. Controlled reproduction of phenomena is the gold standard for the testing of theories and aggregation of data at scale is required to add reliable evidence for social phenomena. Although case studies may illustrate practice and serve as heuristics for theorization, we cannot create enough of them

to fully sample a construct. Yet, in order to claim that anything we talk about is credibly true, we cannot escape the logic of how social research deals with empirical claims. Additionally, as Herbst and Chazan (2023) noted, this way of studying practice empirically is also a way of infusing our theorizing with practice, namely a way to create a feedback loop that not only can add evidence or refute theorizations but also can provide enough agentic variability (many small cases) to stimulate more theorization (see also Herbst et al., in press).

Practical Rationality Resists the Reform Industry

Our research field has historically found sources of research questions in the study of educational reforms championed by our own cadres. The New Math reform in the 1950s and 60s motivated the recruitment of a brilliant generation of scholars to study the research questions the reform raised. Successive reforms have continued to motivate researchers to ask questions that contribute to the growth of our field. Research agendas and projects are sometimes justified with reform proposals—such as, for example, when studies set out to evaluate whether practices of mathematics education reflect what reformers prescribe. Reciprocally, researchers often desire or expect that their research findings will justify recommendations for reform. Although such relationships between research and reform are viable and sometimes useful, research can aspire to do more than prescribe reforms in practice or evaluate whether practice abides by reform recommendations. Furthermore, when researchers conceive of their work as too tightly coupled with a specific reform, they forget that educational reform is also an industry, a sector of the economy, whose prosperity and momentum rely on the constant manufacturing of novelty.

In such a conception of research, researchers can sometimes be assimilated to designers, promoters, or evaluators of reforms. Although it is important for some researchers to study teaching and learning in innovative contexts, we believe that research needs to understand the actual conditions in which teachers work—the reasons why they do what they do and know what they know. We believe that research on mathematics teaching should not assume that any policy change can be viably implemented in an existing education system and instead should recognize that expectations that teachers will be the promoters of radical changes can be exploitive (Osibodu, et al., 2023). We hear teachers say that the constant need for change is part of what makes their work harder to do. Could research in mathematics education also help to moderate the market impulses for constant change?

We believe it is important for researchers to provide concepts that allow teachers to participate in processes of educational reform, not merely by executing what policymakers decide or recommend, but also by responding to such directives through articulating what they know based on their teaching experience and professional judgment. In that sense, practical rationality does not aim to promote or evaluate specific reforms, but to disclose the characteristics of the work of mathematics teaching that any reform will need to build on if it is going to have a chance at being viable.

Improving Instruction: How Practical Rationality Can Help

The title of this presentation asks the question, “What can the theory of practical rationality say about the improvement of mathematics instruction?” Having discussed what practical rationality asserts, the research that it supports, and the motivations behind it, I now want to focus on that question. How can this theory contribute to the improvement of instruction? I want to suggest that the theory has specific uses for four audiences, not necessarily disjoint ones: teacher-researchers, policymakers, teacher educators, and all teachers. They are not prescriptions of how teachers should teach practice, or implications of practice according to the pipeline metaphor, but they are useful and usable connections.

Practical Rationality Supports Teachers' Design Research of Lessons

Earlier, I described my own work as doing research *on* mathematics teaching; but as outcome of that work, practical rationality can support research *in* mathematics teaching—the tinkering and development research of teacher-researchers and other hands-on designers (e.g., design-researchers) who want to improve instruction incrementally.

Practical rationality emphasizes that the continuity of instruction is an important resource that, if maintained and cared for, can be used for improvement. Consider, as an example, the goal of improving instruction by infusing more worthwhile mathematical tasks that require students to think hard to develop new ideas and discuss them with peers (NCTM, 1991). Practical rationality urges designers and teachers to consider the specific instructional exchange in which any such task participates, because the chances for the stable use of such tasks are higher if they serve to teach the content at stake and not just take time that cannot be accounted for by a curricular goal. Furthermore, despite the emphasis that the reform movement puts on being open to many different ways of solving problems, the theory suggests that a teacher needs to be especially attuned to approaches that are serviceable for ushering the content at stake (Herbst et al., 2023).

For example, the geometry task shown in Figure 2 is one that could be used to usher angle relationships, supported by students' use of what they know about the sum of the angles of a triangle which could lead them to make reasoned conjectures about angles formed by transversals crossing parallel lines. A teacher who wanted to use this task would place it around the time when students are learning the properties of alternate interior, alternate exterior, cointerior, coexterior, and corresponding angles. Furthermore, although students could approach the task by measuring angles and counting them, the teacher would be especially attuned to students' approaches that avoid measuring and instead assume some measures and seek to represent the other measures as results of calculation with the known measures. The more deliberate a teacher is in aligning the choice of the work they ask students to do, what they are attuned to see students doing, and the content at stake that such actions help them account for, the more likely it is that the task will become a stable element of the course of studies. Interesting activities that cannot be accounted for by curricular goals run the risk of being dropped because of the institutional obligation to teach the curriculum and make efficient use of time.

Now, the suggestion that the teacher has to be attuned to the strategies that will support such accountability does not mean teachers should discourage any other strategies. A problem like that in Figure 2 could be used as an incremental improvement in a course whose didactical contract is that of *teaching through demonstrating problem types*. In such a context, and especially as introduction to the properties of angles formed by transversals and parallel lines, the problem is unlikely to resemble anything students have done before. How could students relate to the problem? Here is where the concept of instructional situation is useful. Students could frame the activity as something akin to what they have done before and the teacher could certainly support such framing (Herbst et al., 2023). There are two instructional situations that could be considered possible frames for this novel task: first, the situation of *measuring with tools* to which students may have been socialized in elementary school, and second, the situation of *geometric calculation in number* (Hsu & Silver, 2014) to which they may have been socialized in middle school. However, neither of these situations is really instantiated in this activity, the first because the statement of the task suggests that knowing without measuring is desirable, and the second because the diagram does not provide any given measure, markings, or symbols that would suggest calculation to be possible. But if the students could frame the activity as a case of one of those situations, they could get started—the norms of those situations would help them make progress. Framing the task as measurement

would cue the need for a protractor, its correct application to measure, and so on. Framing the task as calculation would help retrieve some known properties (e.g., vertical angles are congruent). Moreover, if the task were done in groups, the chances that different students would frame it using different instructional situations enhances the probability that norms from both situations may support students' activity. This should be seen as positive by the teacher, in that students' sense that they are able to undertake the work in some way makes the task look less like a breach of contract. The teacher, therefore, should expect to see approaches that are characterized by normativity (i.e., they abide by norms of the instructional situations used to frame this novel task) while promoting serviceability (i.e., they stage ideas that support an accounting of the work done after the content at stake). In this sense, multiple solutions will be valuable for the teacher to get their work done—normative ones to get the work going, serviceable ones to move the work toward the content at stake.

Practical rationality identifies the management of these variations among students' work as crucial for a teacher to be able to maintain this problem as a stable element of the course of studies. If a problem were so novel that no prior instructional situation would afford students a place to start, the individual obligation might call on the teacher to provide assistance and this might deprive the students from doing some of the mathematical work (in the limit, this could result in what Brousseau, 1997, called the *Topaze* effect, where all the mathematical work is done by the teacher).

All four obligations can be considered as controls to fine-tune the design of improvements. The interpersonal obligation might modulate how the teacher attends to different students' contributions to the problem, adding considerations to those of normativity and serviceability—for example, the need to make sure to allocate public attention to individual students fairly and not to allow individual students to monopolize attention. The disciplinary obligation should also matter in the management of serviceable work—for example, though the activity supports the conjecture that corresponding angles are congruent and though taking that conjecture as true would facilitate students' making more inferences, the disciplinary obligation should prompt the teacher to ask students whether it is true that corresponding angles are congruent and urge them not to be content with checking by measurement.

This example, I hope, shows that the notions of mathematical work, content at stake, instructional exchange, instructional situation, didactical contract, and professional obligations are useful to design and anticipate the development of lessons that involve more student inquiry than usual in the contract that we called *teaching through demonstrating problem types*. I believe that teachers who want to improve their instruction incrementally can find in practical rationality enough assistance for design and control of their actions to build on prior work and at the same time introduce new forms of mathematical work. The type of work proposed in Figure 2, of the sort “how many measures would we need to make in order to know them all?,” could be utilized with other diagrams and for ushering in other concepts. The format of the task could be used to promote the emergence of theorems or properties and, if used frequently, one could think of this type of task as the basis for an instructional situation of conjecturing. Furthermore, the example shows that although practical rationality provides the means to describe practice, the concepts are sufficiently general to create local prescriptions for the implementation of specific designs and to anticipate observables in such implementation.

Practical Rationality Can Assist Policymakers

Practical rationality can help policymakers understand why some improvement efforts have not worked out well and possibly to design better reforms. A common use I have noted of the professional obligations in research is to examine interviews with practitioners who account for why

things might have not gone according to plan in teaching innovative lessons or implementing innovative practices (e.g., Webel & Platt, 2015). In my opinion, a very regrettable issue with the reform industry in the U.S. is that whereas it relies on institutions to endorse the implementation of reform, it ignores institutional dynamics at the time of supporting teachers in this implementation. Indeed, developers who promote particular curricula or pedagogies often seek the commitment of district and school leaders in implementation, which attests to the developers' recognition of existing institutional structures. Yet, at the time of supporting the implementation by teachers, they seem to ignore or downplay many of the preexisting institutional practices that teachers are still expected to sustain and that will not be easily waived with a commitment to use new materials. Expectations of use of space and time, behavior management, grading, and more are likely to continue to be in place. Reformers who ignore, caricaturize, or minimize those expectations because they are contrary to the reforms they espouse, and who expect that teachers will deal with and resolve those contradictions on their own, are not doing good service to their cause and, in my view, alienating teachers from the reforms.

Reformers who want to see changes taking root need to proceed with a clear understanding of existing institutional practices, accept that these practices serve important purposes (even if they are not the same purposes as the reform), and acknowledge the predicament in which commitments to reform put teachers and leaders. Practical rationality provides a language to model the existing practices against whose backdrop new reforms are pitched. Reforms need to be presented as incremental improvements over those existing practices—not seeking to nullify norms or obligations but seeking to use norms and obligations productively. In particular, reformers should understand that if teachers end up sacrificing some reform practices, it may be for reasons that practical rationality can help identify—and that such reasons, instead of being dismissed as reactionary or incorrect, need to be the source of information for the following cycles of design. Reformers need to realize the importance of stability and control in systems (classrooms, schools, or districts) and acknowledge that teachers and leaders are accountable to maintaining stability and control. Changes that do not anticipate how stability and control will be achieved are more likely to produce stress and backlash than any real instructional improvement.

Practical Rationality Can Guide Mathematics Teacher Educators

There are two ways in which practical rationality can support the work of mathematics teacher educators. Practical rationality can be used to design teacher development interventions that aim to increase professional identity and discourse through collective engagement in tasks of teaching. In StoryCircles (Herbst & Milewski, 2018), which is a technology-mediated, lesson-development process comparable to lesson study, facilitators can use the concepts of practical rationality to bring resources and constraints for teachers to deal with as they virtually teach and revise a lesson (see for example Brown et al., 2021).

Practical rationality can also support mathematics teacher educators as they design the curriculum of mathematics teacher preparation. As a mirror image of my point in the previous section, just as mathematics teachers need a professional discourse with which they can engage with reformers and stakeholders, teacher educators, rather than defaulting to the role of reform propellers, could consider framing their role as teacher advocates. Even though practical rationality could assist policymakers and reformers in making their reforms better, chances are that teachers will continue to have to deal with desires for change from a wide variety of sources, many of whom are ignorant (sometimes willfully so) of the conditions in which mathematics teachers work. Not only education policymakers and reformers, but also ideologues, some of them representing fringe segments of legitimate stakeholders, will continue to feel entitled to expect teachers to subscribe to all sorts of agendas for

change, some of them highly compelling in a general sense but very unclear in the specifics. Some of these ideologues are used to expect teachers to commit to big value statements and convert them magically into the will, the skill, and the power to do things differently in classrooms. Other ideologues, more aware of the role of structural conditions, seem to think that changes to compensation or teacher evaluation mechanisms will incentivize individual teachers to have the will and come up with the skill and the power to increase achievement. It is unlikely that these ideologues will disappear any time soon, and all of them bring to mathematics education practice particular professional or personal biases that prevent them from seeing mathematics teaching practice the way a mathematics teacher sees it. Mathematics teacher educators, particularly those who have had experience as mathematics teachers, could choose to become advocates of their clients and help them gain the knowledge to engage with reformers and defend their practice from ideologues.

Teacher unions often bring up some of the institutional complexities that affect teachers' work to defend the profession against ideologues, but their own ideological bent sometimes prevents them from acknowledging some of the complexities associated with the interpersonal and individual obligations of a mathematics teacher. And almost never are issues of subject matter part of the defense of teaching mounted by teacher unions. Practical rationality can provide concepts and examples for mathematics teachers to participate with stakeholders in negotiations that aim at promoting specific changes. Rather than reducing their defense to the general matters featured in labor contracts, mathematics teachers assisted by their mathematics teacher educator advocates could wield the resources of practical rationality to educate stakeholders, reformers, and ideologues about the predicaments of mathematics teaching.

Indeed, I think this is something that those of us who are not practicing teachers but who understand the work of teaching can help with. As a mathematics education researcher considering the contribution of my research to practice, I realize that rather than becoming an advocate of any ideology, I can exercise my responsibility toward practice in the form of advocating for mathematics teachers: Helping the public understand the conditions and constraints in which mathematics teachers work because those who want mathematics teachers to do things differently need to understand those conditions and work within them. Practical rationality provides a language that can be used to describe the predicaments teachers work under so that the defense of teaching is based not only on the terms of labor law but also on professional considerations.

All Mathematics Teachers Can Use Practical Rationality

I hesitate to say that practical rationality can help teachers reflect about their practice, because almost everything that researchers write is offered to the world with that trite line. Yet, as a former teacher myself, I can attest to the frustration that ambitious teachers often experience—the sense of missed expectations or the sense of not having tried hard enough. If the image from Orwell's *1984* of the horse who kept saying he would try harder in a system that was rigged against him is of any counsel, I think the reflection that practical rationality can support might provide a dose of self-care for teachers. The reform industry will continue to exist, but rather than working hard to meet one reform that goes in one direction and then working even harder to meet another reform that goes in the opposite direction, practical rationality can help teachers examine the specifics of each reform in the context of the obligations they already have. This will allow them to center the long-term continuity of their professional practice and of the institutions that make room for the mathematics instruction of children and youth.

Conclusion: Back to the Relationship Between Research and Practice

At the onset of this essay, I noted that in thinking of the relationship between research and practice, a researcher needs to be aware of a few things. I mentioned, first, the need for awareness of what policymakers and practitioners require. My take is that policymakers in the U.S. require more knowledge about the predicaments a mathematics teacher needs to manage and that teachers require the resources of practical rationality to develop a discourse of professional advocacy with which to interact with policymakers and stakeholders.

I also noted that researchers need to acknowledge the limitations of scholarly knowledge in general and of specific research claims in particular. In this regard, practical rationality is a work in progress—we have shown that it can support descriptive research on mathematics teaching, but we have only scratched the surface of understanding the practice of mathematics teaching. We do not know much about the practice of elementary mathematics teaching, for example. And we are also very clear that our empirical research informs a theoretical model of mathematics teaching practice that, like all models, relies on a reduction of the complexities of actual practices in context. That is, even if our research does show the role that norms and obligations play in rationalizing the decision making of mathematics teachers, we cannot forget how much the theory cannot explain. In particular, it cannot explain how the nuances of individual cases may still owe a lot to chance, idiosyncrasy, serendipity, and also individual will and desire, many of which are ignored as we reduce them to analytic models. The material reasons why individual teachers do things remain to some extent inscrutable, and it would be inappropriate to use practical rationality to justify everything that any individual mathematics teacher does with the complexities the profession needs to manage. As we continue to develop practical rationality, our goal is to understand the work of mathematics teaching in general, not to portray the practice of every individual mathematics teacher. The use that practical rationality has for instructional improvement is therefore prospective and I do not expect or wish for that use to ever be prescriptive to the point of waiving the need for users to apply it judiciously.

I also noted the need to keep in check the researcher's own position, second-guessing how much to infuse their own hopes and values in a transaction of knowledge that can be fraught with power imbalances. Along those lines, I still second-guess the value of explaining the theory to teachers. Some of the tensions and contradictions a mathematics teacher needs to manage are managed subconsciously because time separates spheres of practice (Bourdieu, 1990, p. 86), and I wonder about the extent to which acquiring a theoretical language may impose costs on the plasticity of action.

Finally, I indicated some second thoughts about my consideration of the relationship between practical rationality theory and practitioners here in East Asia, when the context of the development of the theory and its possible applications that I can provide come from the U.S. I have no concrete advice on that matter, other than to recommend that my Asian colleagues distrust any efforts to import recommendations for practice from people unaware of your own institutional and cultural contexts. The way your countries conceive of the role of schooling in society or what it entails to be an individual student in school can be quite different than they are in the U.S. The relationships between research and teaching practice are also a practice, and any adaptation of what I say researchers could do to translate their research to practice needs to be situated in that context.

The connection between research and practice need not be one of providing prescriptions for practice; research and theory can instead provide resources practitioners can use judiciously. The example of practical rationality suggests that research can provide concepts that different types of practitioners (mathematics teachers, policymakers, mathematics teacher educators) may use to control

how they investigate possible improvements in practice and how they explain the outcomes of such attempts.

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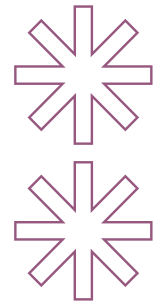
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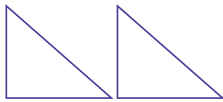
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**The 9th ICMI-East Asia Regional Conference on
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Plenary Panel Discussions



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



Panel Discussion 1

RETHINKING THE ROLE OF MATHEMATICS TEACHERS

Yeap Ban Har

**RETHINKING MATHEMATICS EDUCATION IN THE AGE OF AI:
TOWARD A DIGITALLY TRANSFORMED FUTURE**

Gabriele Kaiser, Mustafa Cevikbas

**DIGITAL TRANSFORMATION IN MATHEMATICS EDUCATION:
RETHINKING CLASSROOM PRACTICES WITH AI AND DIGITAL
TECHNOLOGIES**

Frederick K S Leung

**REIMAGINING MATHEMATICS EDUCATION THROUGH DIGITAL
TRANSFORMATION**

JeongSuk Pang



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



RETHINKING THE ROLE OF MATHEMATICS TEACHERS

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This position paper is an urgent call for teachers to rethink their role in helping all students learn. It is also a call for school leaders and education systems to have frameworks and mechanism to initiate and sustain this transformation in the role of mathematics teachers, and to innovate to catalyze the process. And it is a call for universities to rethink initial teacher preparation as well as teacher professional development to support this transformation. Based on the work with three multi-campus schools in Singapore, Thailand and Indonesia, I will discuss three roles that we need teachers to play in a world where technology is pervasive and rapidly changing. I will also discuss frameworks and mechanisms that these school have instituted to set the process of transformation in motion, along with challenges that they face. I conclude by raising questions for both high-achieving systems as well as low-performing ones, both of which are present in the Southeast Asian region.

Keywords: teachers' role, digital transformation, mathematics learning

Digital Transformation

Digital transformation in education is the process of using digital technologies to fundamentally change how an education system operates, creates products and services, and interacts with stakeholders including students and their parents, teachers and their school leaders, as well as other partners in education. It involves integrating digital technologies into all areas of the education system, leading to a cultural shift that embraces innovation and experimentation.

One possible way to organize our thinking about digital transformation in education is the categories developed by Baker and Smith (2019) in discussing artificial intelligence for education. They proposed examining digital transformation from learner-facing, teacher-facing, and system-facing perspectives. Learner-facing perspectives focus on tools that students use: to receive information and feedback, to collaborate with other students, and to improve their learning. Teacher-facing perspectives focus on tools that help teachers “to reduce their workload, gain insights about students and innovate in their classroom” (Baker & Smith, 2019, p. 12). System-facing perspectives focus on tools that facilitate sharing data with administrators and policymakers for insights about student variables and management processes.

This position paper focuses on a key variable in digital transformation – the teachers. In particular, based on my experience in working with three multi-campus schools in Southeast Asia, I discuss the rethinking of the roles of mathematics teachers in helping students learn in an increasingly technological world. Gladwell (2009) cited the work of Erik Hanushek that estimates that students with a very bad teacher will learn, on average, half a year’s worth of material in one school year while students with a very good teacher will learn a year and a half’s worth of material. Teacher effects are greater than school effects. Teacher effects are stronger than class-size effects. The role of the teacher therefore warrants attention in discussing digital transformation.

Three Roles of the Teacher – Designer, Curator and Facilitator

At Pathlight School in Singapore, a two-campus school that provides primary and secondary mainstream national curriculum for students with autism, driven by a vision of transformative learning, the role of the teacher is redefined to move teachers away from teacher-centred practices such as telling and explaining. The school eventually identified (1) curators of content, (2) facilitators of learning, and (3) designer of learning experiences as the roles of teachers.\

Given the easy availability of content including edited video explanations which can be enhanced with animation, teachers need to be curators of content rather than provider of content and provider of explanations. Curators of content choose the pieces for students to engage with. Curators think about the best filter for students to sieve information and the best lens for students to see with clarity the connectedness of the information.

Given the rapidly changing nature of the job market, students need to learn how to learn. As such, teachers need to be facilitators of learning so that students develop the ability to learn beyond acquisition of content. Facilitators direct students to suitable and suitable amount of resources, not overwhelm them with details. Facilitators tell students the big picture and direct students to resources that they explore and engage with, terminating with different students going away with knowledge that they construct and own. Facilitators scaffold rather than show or tell when students face difficulties.

Given the importance of meaningful learning and traits such as being independent, and these do not lend themselves to frontal teaching that has been conveniently used in many classrooms, teachers need to be designers of learning experiences. Designers have pedagogical practices that are theory-based and research-driven. Designers think about how physical space can be used to set expectations and help students regulate their behaviours during the learning process.

These three roles have been identified to harness the benefits that digital transformation has afforded. In this case, digital transformation goes beyond applying digital technologies in teaching and learning as well as educational management activities. The school focuses on the key player in the equation – the teachers – by defining their role in student learning.

Facilitating Change in How Teachers See Their Role

With the role of the teacher defined, it is necessary to put in place a mechanism to disrupt inertia among some teachers who prefer to teach by telling or by explaining. When there is a change, teachers respond in four ways.

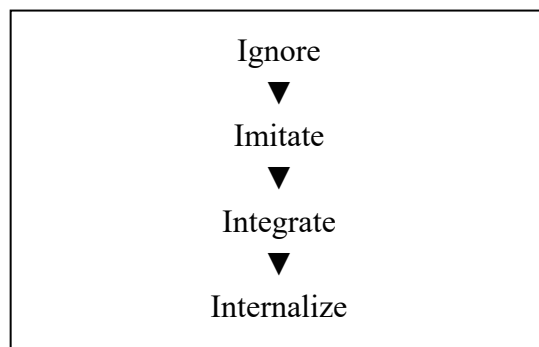


Figure 1. 4-I model of teacher change

Teachers at the Ignore level continue with classroom practices that they have inherited from their own teachers and those that they have become comfortable with – telling and explaining, with little regards for students’ actual learning. They essentially ignored the defined role in favour of what they have become accustomed to. Teachers at the Imitate level makes an unsuccessful attempt at the defined role. They may imitate some superficial features of the defined role but are unable to carry out the role in most other instances. Teachers at the Integrate level are successful in carrying out the defined roles, just like those at the Internalise level. However, teachers at Internalise level have fundamental shift in their belief system about teaching and learning.

At Anglo Singapore International School in Thailand, a three-campus K-12 school in two cities that provides international students with a juxtaposition of Singapore and British curricula, the mechanism to move teachers towards the last two levels is a theory-driven framework, which they refer to as high-quality learning framework. This mechanism has two parts.

A key feature of this framework is the identification of three outcomes – (1) meaningful learning, (2) independence among students, and (3) dignified learning atmosphere. Learning is deemed to have taken place if these outcomes are observed. Having the end in mind (Covey, 2004) articulated compels teachers to make the shift.

The framework also delineating learning experiences that teachers should provide for their students. The six learning experiences are (1) exploring, (2) discussing, (3) consolidating, (4) writing (journal), (5) reading, and (6) reflecting, generated based on learning theories. Providing these learning experiences on regular basis increases the chances of teachers making the shift.

At Pahoa School in Indonesia, a two-campus K-12 school with a sister school in a different city that provides local students with the national curriculum tempered with an international perspective, digital transformation takes place in the form of adopting technology that allows teachers to monitor closely the progress of individual students, ensuring that even the weakest learners received targeted help to master at least essential concepts.

The mechanism the school uses to help teachers make the shift is an auditing platform where teachers take turns to report their experiences in carrying out the defined roles with their peers providing. This sharing and feedback platform which they call Audire – Latin for to hear – accelerates teachers assuming the defined roles.

Concluding Remarks

In this paper, I make a call to rethink the role of teachers in helping all students learn mathematics. No amount of technology can make learning more effective and the work of teaching more efficient unless teachers see their role as helping students construct understanding rather than acquiring facts and formulae.

In Southeast Asia, we have high-performing systems as well as those with low average achievement scores. For low-performing systems, the question is what the order of priority for the necessary actions is. We know what needs to be done. However, we need to also know what is the first thing that needs to be done? I conjecture that the answer has something to do with teachers. For high-performing systems, the question is what is the motivation for the shift, given that the achievement level is already high?

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RETHINKING MATHEMATICS EDUCATION IN THE AGE OF AI: TOWARD A DIGITALLY TRANSFORMED FUTURE

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This paper explores the role of artificial intelligence (AI) in mathematics education, emphasizing its potential to transform teaching, learning, and assessment, while addressing emerging concerns related to its implementation and impact. Moving beyond tool integration, digital transformation entails a systemic shift toward student-centered, adaptive learning environments. AI-driven platforms enable personalized instruction, real-time feedback, and differentiated assessment, while immersive technologies support deeper understanding. The teacher's role evolves into that of a facilitator and data-informed decision-maker. Despite these advances, challenges persist, including inequitable access, teacher preparedness, and ethical considerations. Addressing these issues is critical to ensuring inclusive and impactful digital transformation. The paper concludes with future-oriented insights, predicting a mathematics education landscape deeply integrated with computational thinking, interdisciplinary learning, and sophisticated AI applications. Thoughtful implementation will be key to transforming mathematics education for diverse learners in a rapidly changing world.

Keywords: artificial intelligence, digital technology, mathematics education

Introduction

Digital transformation in mathematics education represents more than the simple integration of digital tools into classroom settings; it reflects a comprehensive shift in pedagogical practice. It embodies a fundamental paradigm shift in the ways teaching, learning, and assessment are conceptualized and enacted, driven by rapid advancements in technologies such as artificial intelligence (AI), machine learning, virtual and augmented reality (VR/AR), and data analytics. This transformation represents a move toward more personalized, interactive, and contextually relevant learning experiences, with profound implications for educators, learners, and educational institutions.

This paper focuses on the conceptualization of digital transformation within mathematics education. It critically examines how AI and digital technologies are reshaping curriculum design, instructional practices, and assessment strategies. Furthermore, it discusses the pressing challenges and future trajectories emerging in the evolving AI-driven educational landscape.

Digital Transformation in Mathematics Education

Digital transformation in mathematics education entails a profound reimagining of pedagogical structures and learning environments through the integration of digital innovation. It moves beyond the superficial adoption of digital tools to encompass fundamental changes in instructional design, content delivery, and assessment practices (Voogt et al., 2015). At its core, this transformation signifies a shift from traditional teacher-centered models toward student-centered learning environments, supported by adaptive technologies that respond to individual learners' trajectories and needs.

Central to this shift is the creation of responsive learning environments that leverage technologies like AI to provide real-time feedback, personalize instructional pathways, and foster immersive learning experiences (Luckin et al., 2016). Dynamic simulations, for instance, allow students to visualize and manipulate mathematical models, facilitating deeper conceptual understanding that traditional methods often fail to achieve (Holmes et al., 2019). This process reflects not only a technological change but also an epistemological and practical evolution in how mathematics is conceptualized, taught, and assessed (Cevikbas & Cevikbas, 2015). A widely cited example is the ALEKS (Assessment and Learning in Knowledge Spaces) intelligent tutoring system, which continuously diagnoses student knowledge in mathematics and dynamically adjusts learning pathways based on mastery, thereby promoting personalized and self-regulated learning experiences (Falmagne et al., 2013).

AI and Digital Technologies in Classroom Practice

AI and other digital tools are increasingly used to customize educational experiences. In lesson planning and curriculum design, AI-powered adaptive learning systems adjust content difficulty in response to student performance data, helping educators identify learning gaps and respond accordingly (Kulik & Fletcher, 2016). These systems support differentiated instruction and provide pathways for all learners to engage with mathematical content meaningfully. A prominent example is the ASSISTments platform, which enables teachers to assign online mathematics homework while receiving detailed reports on student performance, allowing for immediate instructional adjustments based on learning analytics (Heffernan & Heffernan, 2014).

Assessment practices are likewise undergoing a transformation. Traditional summative assessments are giving way to formative, analytics-driven approaches that provide continuous insights into students' progress (Knight, 2020). In particular, Stealth Assessment, embedded within digital learning environments such as games or simulations (Shute & Ventura, 2013), can capture learners' problem-solving processes without interrupting the learning experience, offering a deeper and more dynamic understanding of students' mathematical thinking. In this landscape, the teacher's role evolves from content deliverer to learning designer and facilitator. Teachers interpret learning analytics, guide students in navigating digital environments, and ensure equity in access and participation. The interplay between teacher agency and AI capabilities supports pedagogical practices that are both scalable and personalized (Holmes et al., 2019).

Emerging technologies like VR/AR provide spatial and interactive representations of mathematical ideas, enhancing learners' ability to grasp abstract concepts. AR applications can overlay geometric constructions on physical space, while VR environments immerse students in mathematical problem-solving scenarios (Cevikbas et al., 2023). For instance, CoSpaces Edu enables students to create and explore 3D mathematical models, offering hands-on opportunities to engage with spatial reasoning and geometric concepts. Such AR/VR experiences foster engagement and support diverse learners through multimodal representations of knowledge (Akman & Cakir, 2020).

Challenges and Future Directions

Despite the potential of digital transformation, which has just been elaborated, significant challenges remain. One major concern is the digital divide. Unequal access to technology and internet connectivity exacerbates educational inequities, particularly for students in marginalized or remote communities (OECD, 2021). Addressing this issue requires systemic efforts to ensure infrastructure, devices, and technical support are available to all learners.

Another challenge lies in teacher readiness. The pace of technological advancement often outstrips teacher training and professional development programs. Teachers need sustained, context-sensitive support to integrate digital tools effectively into their practice (Ertmer & Ottenbreit-Leftwich, 2010). Without such support, the benefits of technology risk remaining untapped.

Furthermore, ethical and pedagogical concerns must be addressed. While AI can tailor learning experiences, it cannot replicate the human elements of empathy, motivation, and relational understanding essential to effective teaching. For example, research has shown that predictive models used in learning analytics (such as early warning systems) can inadvertently reinforce existing biases if not properly monitored (Slade & Prinsloo, 2013). Ensuring that AI-driven technologies amplify rather than replace the relational aspects of teaching remains a critical priority.

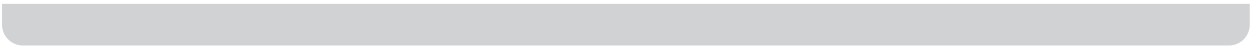
Looking forward, the future of mathematics education will be marked by increasing personalization, interdisciplinary learning, and the integration of computational thinking. By 2030, AI systems may enable real-time mapping of individual learners' cognitive pathways, allowing for precision learning experiences tailored to evolving competencies. Mathematics education will likely merge more closely with data science, algorithmic reasoning, and real-world problem-solving (Weintrop et al., 2016).

To harness this potential, educational stakeholders must prioritize equitable access, ethical use, and teacher empowerment. Only through such deliberate efforts can digital transformation in mathematics education fulfill its promise of creating inclusive, meaningful, and future-ready learning environments for all students.

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DIGITAL TRANSFORMATION IN MATHEMATICS EDUCATION: RETHINKING CLASSROOM PRACTICES WITH AI AND DIGITAL TECHNOLOGIES

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The rapid advancement of digital technologies and artificial intelligence (AI) is driving a significant transformation in the landscape of education, particularly within the realm of mathematics. This paper explores the concept of digital transformation in mathematics education, emphasizing the integration of AI and digital tools to enhance teaching methodologies, learning experiences, and educational outcomes. These technologies offer new possibilities for curriculum design, assessment, and the evolving role of teachers. The paper discusses how AI-driven platforms and digital resources facilitate personalized learning, foster deeper mathematical understanding, and promote student engagement through interactive simulations and real-world applications. Despite the promising benefits, challenges such as the digital divide and the need for professional development for educators remain critical. The paper concludes by outlining future directions, including ensuring equitable access to technology, investing in teacher training, and continuously innovating to keep pace with technological advancements.

Keywords: digital technologies, artificial intelligence (AI), digital transformation in mathematics education, challenges of digital transformation, integration of AI in mathematics teaching

Introduction

The landscape of education is undergoing a significant transformation, driven by the rapid advancement of digital technologies and artificial intelligence (AI) (Selwyn, N., 2016; Fullan, M., & Langworthy, M., 2014; Luckin, R. et al, 2016; Holmes, W. et al, 2019). In the realm of mathematics education, digital transformation is not merely about integrating technology into the classroom; it involves a fundamental shift in teaching methodologies, learning experiences, and educational outcomes (Hoyle, C., & Lagrange, J. B., 2009). This paper explores the concept of digital transformation in mathematics education, examines how AI and digital tools are reshaping classroom practices, and discusses the challenges and future directions of this transformation.

Defining Digital Transformation in Mathematics Education

Digital transformation in mathematics education refers to the integration of digital technologies into teaching and learning processes to enhance educational outcomes. This transformation involves the adoption of digital tools and resources to facilitate learning, improve engagement, and personalize education. It encompasses a wide range of technologies, including online learning platforms, educational software, interactive simulations, and AI-driven tools that support both teachers and students in the learning process (Drijvers, P., 2015).

Digital transformation aims to move beyond traditional teaching methods, which often rely heavily on rote memorization and passive learning. Instead, it emphasizes active learning, critical thinking, and problem-solving skills. In mathematics education, this transformation can lead to more dynamic and interactive classrooms where students engage with mathematical concepts through digital simulations, collaborative tools, and real-world applications.

AI and Digital Technologies in Practice

Reshaping Classroom Practices

AI and digital tools are revolutionizing the way mathematics is taught and learned in classrooms. These technologies offer new possibilities for curriculum and lesson design, assessment, and the teacher's role in the learning process.

Curriculum and Lesson Design:

Digital tools enable educators to create interactive and personalized learning experiences. For example, platforms like dynamic geometry environments, GeoGebra and Desmos allow students to explore mathematical concepts through dynamic visualizations and simulations (Hegedus, S., & Moreno-Armella, L., 2010). These tools help students understand abstract concepts by providing visual representations and interactive elements that make learning more engaging.

AI-driven platforms can also analyze student performance data to tailor lessons to individual learning needs. This personalization ensures that students receive the appropriate level of challenge and support, enhancing their understanding and retention of mathematical concepts.

Assessment:

AI technologies are transforming assessment practices by providing real-time feedback and adaptive testing. For example, adaptive testing is already used in PISA to improve the accuracy of measurements of student ability at the ends of the score distribution (i.e., high- or low-performing groups of students) (OECD, 2023). AI-powered assessment tools can evaluate student responses, identify areas of weakness, and suggest targeted interventions. This immediate feedback loop helps students understand their mistakes and learn from them, promoting a deeper understanding of mathematical concepts.

Moreover, digital assessments can include interactive elements such as simulations and problem-solving tasks that assess higher-order thinking skills. This shift from traditional paper-based tests to digital assessments aligns with the goals of digital transformation by emphasizing critical thinking and problem-solving.

The Teacher's Role:

The role of teachers in the AI era is evolving from being the primary source of information to facilitators of learning. With AI and digital tools handling routine tasks such as grading and data analysis, teachers can focus more on guiding students, fostering critical thinking, and supporting collaborative learning.

Teachers can use AI-generated insights to identify students who need additional support and tailor their teaching strategies accordingly. This shift allows teachers to spend more time on meaningful interactions with students, enhancing the overall learning experience.

Fostering Deeper Mathematical Understanding and Engagement

AI and digital technologies have the potential to foster deeper mathematical understanding and engagement among diverse learners. By providing personalized learning experiences, these tools can address individual learning needs and preferences, making mathematics more accessible and enjoyable for all students.

Interactive simulations and visualizations help students grasp complex concepts by allowing them to experiment and explore in a risk-free environment. Collaborative tools enable students to work

together on problem-solving tasks, promoting communication and teamwork skills.

Moreover, digital platforms can connect students with real-world applications of mathematics, demonstrating the relevance and importance of mathematical concepts in everyday life. This contextualization enhances student engagement and motivation to learn.

Current State and Future Directions

Challenges in Digital Transformation

Despite the potential benefits, digital transformation in mathematics education faces several challenges. One of the main challenges is the digital divide, which refers to the gap between those who have access to digital technologies and those who do not (Zhao, Y., & Watterston, J., 2021). Ensuring equitable access to technology and internet connectivity is crucial for successful digital transformation (Warschauer, M., & Matuchniak, T., 2010).

Another challenge is the need for professional development and support for teachers (Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.), 2014). Educators must be equipped with the skills and knowledge to effectively integrate digital tools into their teaching practices. Ongoing training and support are essential to help teachers adapt to new technologies and pedagogical approaches.

Future Directions and Priorities

As digital transformation continues to shape mathematics education, several key priorities and predictions emerge for the future:

Equity and Access:

Ensuring equitable access to digital technologies and resources for all students, regardless of their socio-economic background, is a top priority. This includes providing necessary infrastructure, devices, and internet connectivity.

Professional Development:

Investing in professional development for teachers is essential to equip them with the skills needed to effectively integrate digital tools into their teaching practices. This includes training in using AI-driven platforms, designing interactive lessons, and leveraging data for personalized learning.

Integration of AI:

The integration of AI into mathematics education will continue to evolve, with a focus on developing more sophisticated AI-driven tools that can provide personalized learning experiences, real-time feedback, and advanced analytics.

Collaboration and Communication:

Digital transformation will emphasize the importance of collaboration and communication skills. Digital tools will facilitate collaborative learning experiences, enabling students to work together on problem-solving tasks and projects.

Real-World Applications:

Mathematics education will increasingly focus on real-world applications, using digital tools to connect mathematical concepts to everyday life and various industries. This approach will enhance student engagement and motivation to learn.

Continuous Innovation:

The field of digital transformation in education is rapidly evolving, and continuous innovation will be necessary to keep pace with technological advancements. Educators and institutions must remain adaptable and open to new approaches and tools.

Mathematics Teaching and Learning in the Classroom:

In the context of mathematics teaching and learning in the classroom, the advent of generative AI, such as ChatGPT, which can solve most, if not all, of the mathematical problems students encounter in school — complete with detailed solution steps and clear explanations of the reasoning — raises important questions. What mathematics should students learn in this AI era, and how should it be taught in schools? This issue deserves attention and should be a focus for future research and exploration by educators (Zawacki-Richter, O. et al, 2019).

One significant implication is that, since students can easily obtain solutions to mathematical problems using generative AI — and it is challenging for teachers to determine whether a student solved the problem independently or with AI assistance — greater emphasis should be placed on evaluating the appropriateness and accuracy of solutions rather than on memorizing mathematical facts or practicing calculation skills. When generative AI provides a solution involving mathematical concepts that students have not yet learned, teachers should guide students in acquiring this new knowledge using online resources. This approach aligns with problem-based learning (PBL), leveraging AI, digital technologies, and large language models (LLMs).

Conclusion

Digital transformation in mathematics education offers exciting opportunities to enhance teaching and learning experiences. By leveraging AI and digital technologies, educators can create interactive, personalized, and engaging learning environments that foster deeper mathematical understanding and critical thinking skills. However, addressing challenges such as the digital divide and providing ongoing professional development for teachers are crucial for successful implementation. As we look to the future, prioritizing equity, access, and continuous innovation will be key to realizing the full potential of digital transformation in mathematics education.

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REIMAGINING MATHEMATICS EDUCATION THROUGH DIGITAL TRANSFORMATION

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This paper explores the meaning and implications of digital transformation in mathematics education, focusing on artificial intelligence (AI) and emerging digital technologies. It investigates how AI and digital tools reshape classroom practices, including teaching, learning, assessment, and the evolving roles of teachers and students. Digital transformation goes beyond the mere adoption of tools; it entails a reconfiguration of epistemologies, pedagogical practices, and the relationships among teachers, students, and resources. This paper identifies the key challenges and priorities for the future of mathematics education in the AI era: curricular design with critical AI literacy, equity and ethical concerns, teacher professional development, and human agency.

Keywords: digital transformation, teacher role, human agency, professional development, adaptability

Introduction

This paper is organized around three given questions: (a) What does “digital transformation” in mathematics education entail from your perspective? (defining digital transformation); (b) How are AI and digital tools reshaping classroom practices, including curriculum or lesson design, assessment, and the teacher’s role? How can these technologies foster deeper mathematical understanding and engagement among diverse learners? (AI and digital technologies in practice); and (c) What are the main challenges you are addressing in digital transformation? What key priorities or predictions do you have for the future of mathematics education in the AI era? (current state and future directions)

Digital Transformation in Mathematics Education

Digital transformation in mathematics education represents a profound shift in teaching, learning, and assessment, driven by AI and emerging digital technologies. This includes integrating advanced tools such as AI-driven adaptive learning platforms, interactive simulations, and dynamic visualization software, enabling personalized and differentiated instruction. These tools not only replace traditional media, but also reshape how mathematical concepts are constructed, perceived, and justified. Pedagogical practices are increasingly shaped by the affordances and limitations of digital environments. Digital transformation necessitates a fundamental reimagining of teaching and learning mathematics, far beyond just integrating technology or using digital tools as additions to traditional methods. In this sense, digital transformation should be seen as a fundamental shift in knowledge production, classroom dynamics, and the relationships between humans and technology (Ouyang & Jiao, 2021; Sinclair et al., 2022).

AI and Digital Technologies in Practice: Teaching, Learning, and Assessment

The recent rise of AI and digital technologies has added layers of complexity to the digital transformation of classroom practice. These technologies influence teaching, learning, and assessment,

calling for new roles for teachers and students in digital environments (Drijvers & Sinclair, 2024; Weigand et al., 2024).

Teaching and Teacher Reconsidered: Toward Critical Practitioners

AI technologies assist teachers in developing tailored curricula and lesson plans that address individual student needs, enabling greater differentiation in instruction. When designing lessons, teachers interact with various digital resources, including digital textbooks, digital manipulatives, and adaptive platforms. Unlike traditional materials, digital resources offer dynamic, interactive, and adaptable content. However, the effectiveness of these resources depends on their quality, alignment with curriculum standards, and user-friendliness. Teachers' ability to interpret and integrate these tools into their instruction is crucial (Trgalová & Tabach, 2024).

During lessons, AI-driven analytics support real-time instructional decisions, offering immediate insights into student performance and allowing teachers to adjust instruction to address learning gaps or misconceptions. This shifts teachers' roles from content providers to active facilitators guiding students' inquiry and exploration. Teachers become orchestrators of mathematical discourse, critics of algorithmic feedback with system-generated data, and co-designers of learner-AI interaction. Teachers increasingly help students navigate and question AI-generated explanations, promoting metacognitive engagement and critical thinking.

AI tools also support teachers in reflecting on their teaching effectiveness. By analyzing data collected during lessons, teachers can better understand student progress, identify strengths and weaknesses, and refine future strategies. This reflective practice becomes more informed and data-driven, providing teachers with insights into their professional growth and the continuous improvement of instructional quality. Trgalová and Tabach (2024) highlight the growing demand for teachers to act as designers of digital resources. Their *digital resource design capacity* model outlines the phases of planning, in-the-moment adaptation, and post-lesson redesign, underscoring the situated and iterative nature of teaching with technology. Similarly, Geraniou et al. (2024) introduce the construct of *mathematical digital competency for teaching*, integrating pedagogical design, orchestration, and documentation practices to foster a holistic approach to digital teaching competence. Dilling et al. (2024) suggest *the media, pedagogy, and content model*, emphasizing teachers' ability to evaluate, select, and use digital technologies for effective teaching. Despite their different theoretical backgrounds, these models collectively underscore the evolving role of teachers as critical practitioners, responsible for applying technologies that enhance the learning process and foster more effective teaching practices.

Learning and Learner Reconsidered: Toward AI-Empowered Agency

Mathematics instruction traditionally involves diagnosing students' understanding, teaching uniform content, and providing remedial or enrichment support. AI and digital tools enable fully personalized learning pathways, tailored to each student's unique needs, abilities, and preferences. Even within the same mathematical concept, such tools can adjust the complexity, representations, and context for diverse learners. In Korea, the growing integration of AI in elementary and secondary mathematics education demonstrates the potential for personalized learning. Research has focused on the effectiveness of AI-supported learning tools, which cater to students' cognitive needs and provide individualized learning support (Park & Lim, 2023).

Pepin et al. (2025) highlight that AI technologies like ChatGPT offer personalized learning experiences, helping students grasp mathematical concepts at their own pace and providing real-time feedback. This personal engagement with AI tools enhances student agency, fostering greater au-

tonomy, and reflective thinking. AI tools also fundamentally transform the experiences and processes of learning, where the learner is no longer a passive recipient but a co-constructor of knowledge in dynamic interaction with AI systems. In this sense, personalized learning does not imply passive adherence to prescribed paths. Students actively interact with AI technologies and digital resources, with teachers providing professional support and guidance to help students effectively navigate their unique learning journeys. This active interaction resonates with the humans-with-media perspective (Engelbrecht & Borba, 2024), which emphasizes mathematical learning as emerging dynamically from collectives formed by students collaboratively engaging with digital technologies. From this viewpoint, AI tools are not simply resources or assistants; they actively shape and are shaped by learners' mathematical reasoning and knowledge construction processes.

As learning environments continue to evolve, students can move beyond collaboration to take leadership in their learning process, actively shaping learning goals, evaluating algorithmic feedback, and making informed decisions based on AI-generated insights. This trajectory aligns with what Ouyang and Jiao (2021) define as the *AI-empowered* paradigm, in which learners assume agency as leaders—setting directions, questioning recommendations, and becoming co-designers of their AI-supported learning paths. This vision of learner-as-leader represents a significant pedagogical shift, where mathematical learning is no longer delivered to students, but co-created through reciprocal, reflective, and adaptive human-AI partnerships.

A related question is whether complete individualization could result in entirely distinct curricula for each student, raising the need for thoughtful structuring of foundational knowledge versus specialized learning. For example, mathematics curriculum might maintain common core learning until middle school, after which learning paths diverge based on students' interests and career aspirations. Additionally, AI-driven personalized learning should aim beyond mere knowledge acquisition; it must strengthen mathematical competencies such as problem-solving, reasoning, communication, and collaboration. This requires carefully designed interactions and tasks that not only facilitate conceptual understanding but also support the holistic development of mathematical proficiency.

Assessment Reimagined: Toward Dynamic and Reflective Assessment

AI and digital tools provide effective means of evaluating student understanding, particularly for objective question types such as multiple-choice or short answers. These automated assessments reduce teachers' workloads of continuous monitoring and provide real-time analytics on student comprehension, informing instructional decisions during lessons (Weigand et al., 2024). As Cusi and Morselli (2024) emphasize, the teacher's role in formative assessment goes beyond facilitation; it involves empowering students to reflect on their learning processes, using digital tools to visualize and discuss responses, thereby promoting deeper engagement. In this sense, assessment becomes an ongoing, dynamic process integrated into instruction rather than isolated events.

However, assessing higher-order thinking and complex problem-solving abilities remains challenging with current AI technologies. Teachers play a crucial role in supplementing AI-driven assessments, focusing on evaluating qualitative aspects of student responses, facilitating deeper discussions, and guiding students through complex problem-solving processes. As assessments become more data-rich and immediate, the role of the student also evolves: students become more active in their assessment processes, using formative feedback provided by AI tools and teachers to self-regulate and adjust their learning strategies.

Challenges and Future Directions

Despite the affordances of digital and AI-based technologies, several challenges persist. These challenges highlight the need for systemic approaches that go beyond mere tool implementation.

Curricular Design for Promoting Reasoning and Critical AI Literacy

In the AI era, mathematics curricula must rethink what constitutes essential mathematical knowledge and competencies. As AI technologies become ubiquitous, mathematics education should prioritize content and skills that are uniquely human or significantly enhanced by human-AI interaction. This includes deeper conceptual understanding, critical thinking, reasoning, effective communication, and ethical interpretation of mathematical information.

AI-enabled individualized learning pathways, while beneficial, risk oversimplifying students' thinking patterns, especially when AI assessments label learners as proficient or deficient in certain areas. This issue is particularly acute at elementary levels, limiting students' critical perspectives and fostering fixed mindsets. AI-generated explanations are often presented with high confidence regardless of correctness, raising concerns about overreliance, reduced student agency, and the erosion of epistemic vigilance (Pepin et al., 2025). The uncritical adoption of AI systems without pedagogical grounding may reduce learning to task completion, neglecting deeper engagement with human oversight. Educators must prepare students to critically evaluate algorithmic outputs and to understand the limitations and assumptions behind such systems. Consequently, mathematics curricula must promote mathematical reasoning and critical AI literacy.

To ensure the development of high-quality mathematics curricula appropriate for the AI era, continuous evaluation of curricular effectiveness through empirical research and practical classroom implementations is essential. Such curricula must facilitate the holistic development of mathematical competencies, actively engage learners in critical thinking about mathematical applications and data usage, and adequately prepare students for effective and informed interaction with AI technologies in their future academic and professional lives.

Equity and Ethical Concerns for Inclusive and Responsible AI Integration

While the core promise of AI applications in mathematics education lies in personalized and adaptive learning experiences for individual students, significant equity and access challenges persist. In contexts like Korea, where most schools have access to digital infrastructure, equity concerns center more on the *quality* of interaction with AI systems than on access. AI-driven personalization may unintentionally widen achievement gaps, advancing high-performing students while confining others to repetitive basic tasks. Equity in digital technology use remains under-researched and requires deliberate design to address access, representation, and responsiveness to diverse learners (Drijvers & Sinclair, 2024). To address this, educators should regularly review AI-generated learning paths, ensuring opportunities for upward mobility and cross-level engagement. Assessment strategies should also combine AI analytics with teacher judgment to preserve flexibility and support all learners' growth.

Additionally, privacy concerns arise from the extensive personal data required by AI systems to deliver personalized experiences. Ensuring the ethical collection, protection, and responsible use of student data becomes critical, necessitating transparent policies and robust digital literacy education, as emphasized in recent policy guidance (Korean Ministry of Education, 2022; U.S. Department of Education, 2023), which highlights the importance of fairness, bias mitigation, and privacy in AI-supported learning environments. Addressing these challenges involves concerted efforts to

provide equitable technological resources, develop policies protecting student privacy, and ensure inclusive access to AI-enhanced educational opportunities.

Professional Development for AI-Ready and Critically Engaged Teachers

The successful integration of AI into mathematics education is deeply influenced by teachers' perceptions and readiness to adopt new technologies (Yeo et al., 2024). Therefore, teacher education programs should begin with pre-service teachers to guide them to be ethically aware and digitally fluent. Teachers also require robust support systems that include continuous, practice-oriented training opportunities and access to high-quality resources that align with real-world classroom contexts. Professional development should not only build technical proficiency but also foster the positive attitudes for successful AI adoption in mathematics education. As Li (2024) points out, teachers' positive attitudes, supported by comprehensive TPACK, foster their willingness to integrate AI tools into their classrooms. Similarly, Lee et al. (2024) found that teachers' positive attitude toward using AI mathematics learning system, with the support of perceived usefulness for mathematics learning and perceived ease of use of AI, significantly influence their intention to use AI. In this regard, recent challenges related to the introduction of AI-based digital textbooks in the Korean context illustrate that top-down, nationally mandated policies may not be effective if they do not genuinely reflect teachers' perspectives, readiness, and needs (Kim et al, 2024).

To engage meaningfully in AI-empowered pedagogies, teachers require new forms of professional preparation. This includes not only technical fluency but also a capacity to orchestrate AI-mediated interactions, interpret complex feedback, and guide students through human-AI co-learning processes. Effective professional development must address this expertise explicitly, while fostering teachers' critical engagement with AI systems, and ethical considerations around data privacy.

Conclusion: Reasserting Human Agency in AI-Enhanced Mathematics Learning

Digital transformation in mathematics education is not merely a technological evolution but a shift in the very foundations of teaching and learning. It requires educators, researchers, and policymakers to deeply consider the epistemological commitments and pedagogical priorities underlying technology integration. AI should be seen not as a monolith, but as a continuum of practices shaped by diverse perspectives, interactions, and degrees of human agency.

The future of mathematics education must carefully examine not only what technologies can achieve, but how these technologies are designed and implemented. Central to this reconfiguration is that knowledge is generated through the distributed interactions among learners, tools, platforms, homes, and AI agents (Engelbrecht & Borba, 2024). Continuous dialogue within the mathematics education community is crucial to navigate these complexities ethically and effectively, ensuring that digital transformation genuinely enhances educational outcomes and equity.

Future research should examine how AI can be positioned not as a replacement for human judgment but as a mediator of mathematical meaning—supporting students in posing questions, constructing explanations, and engaging in collaborative inquiry. AI needs to become a dialogic partner rather than a source of truth, while humans remain active constructors of mathematical meaning. This aligns with the paradigm of AI-empowered learning, where the student is no longer guided solely by the system, but becomes an epistemic agent who negotiates, adapts, and leads within a complex learning ecosystem (Ouyang & Jiao, 2021). In this sense, the future of mathematics education lies not only in leveraging advanced technologies but in reasserting the humanistic and interpretive dimensions of mathematics in digitally mediated worlds.

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Panel Discussion 2

**MATHEMATICS EDUCATION IN SINGAPORE
– AN INSIDER’S PERSPECTIVE**

Berinderjeet Kaur

**KOREAN MATHEMATICS EDUCATION AT A CROSSROADS:
BALANCING TRADITION, INNOVATION, AND EQUITY**

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**THE CHANGING LANDSCAPE OF MATHEMATICS EDUCATION IN
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**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



MATHEMATICS EDUCATION IN SINGAPORE – AN INSIDER’S PERSPECTIVE

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This position paper on Mathematics Education in Singapore is limited in scope to school mathematics. The paper elaborates four key education policies / initiatives and the setting up of the Centre for Research in Pedagogy and Practice (CRPP) at the National Institute of Education (NIE) in 2002, that have shaped Mathematics Education in Singapore from the late 1950's to the present. Two of the education policies / initiatives, the white paper of 1956 and 'towards excellence in schools' report of 1987, have had direct bearings on the intended school mathematics curriculum. Whilst another two, Thinking Schools, Learning Nation (TSLN) of 1997 and Teach Less, Learn More (TSLN) of 2005, have had significant impact on mathematics teacher development. From 2002 to the present, CRPP's research activities have supported professional development of mathematics teachers and generated findings that have fueled research informed mathematics teaching and learning in Singapore schools. Befittingly, in the present digital era, research agenda for mathematics education in Singapore must address the needs of 'digital literate global citizens of the future'.

Keywords: Mathematics education, mathematics teacher, school mathematics, research, Singapore.

Introduction

This paper contributes to the theme “The Evolution of Mathematics Education in Asia” through an exploration of policy curriculum that influenced the development of mathematics education in Singapore. The policy curriculum “consists of educational policies and discourses at the intersection between schooling, culture, and society, embodying a conception or paradigm of what schooling should be with respect to the society and culture” (Deng et al., 2013, p. 6). It shapes the classroom curriculum, which is what is taught and learned in the classroom, via the programmatic curriculum. The programmatic curriculum “translates the ideals and expectations embodied in the policy curriculum” (p. 7) and “constitutes an organizational and operational structure within which classroom practice takes place” (p. 7). In this paper, I will elaborate four key education policies / initiatives and the setting up of the Centre for Research in Pedagogy and Practice (CRPP) at the National Institute of Education (NIE) in 2002, that have shaped Mathematics Education in Singapore from the late 1950's to the present. Following which, in the context of globalization and digital transformation, I will speculate the challenges ahead of mathematics educators in Singapore.

Singapore is an island, with an area of 719.1 square kilometres. The population is approximately 5.9 million and the GDP per capita as of December 2024 was US Dollars \$90, 689. In rank order, the top three budget items of the government expenditure are Defence, Health and Education respectively. As people are the only natural resource of Singapore, the nation spares no effort to actualize its simple objective of education that is:

... to educate a child to bring out his greatest potential so that he will grow into a good man and a useful citizen (Lee, 1979, p. iii)

In Singapore, education is also a key enabler of social mobility. The system provides equal opportunity for every child by i) ensuring that no child is deprived of educational opportunities because of their financial situations; ii) leveraging on the school system to provide more support for families from poorer backgrounds; iii) investing in pre-school education targeted at children with families

with poorer backgrounds; and iv) investing in levelling up programmes in primary schools that attempt to level up academically weaker students in both English and Mathematics, so as to improve their foundations for future learning (Heng, 2012).

Two Key Education Policies and Their Impact on the Intended School Mathematics Curriculum

The White Paper of 1956 (Yip et al., 1990)

In 1959, when the People's Action Party (PAP) came to power, it acted upon the White Paper of 1956 and put in place a Five-Year Plan in education. The main features of this plan were:

1. Equal treatment of the four language streams of education: Malay, Chinese, Tamil and English;
2. The establishment of Malay as a national language of the new state;
3. Emphasis on the study of **Mathematics**, Science and Technical Subjects.

The White Paper was the impetus for coherent mathematics content to be taught in school leading to national examinations at grades 6, 10 and 12 from 1960 onwards. The goal of such examinations was to ensure that all students underwent common experiences in school with respect to the curricula that was taught (Kaur, 2009a). The first Ministry of Education (MOE) designated mathematics syllabuses for primary and secondary schools were available for use in Singapore schools in 1960. A revision of this set of syllabuses for both the primary and secondary schools took place in the late 1960s in response to the global "Math Reform of the 1960s". Over the years, several revisions of the syllabuses have taken place (Kaur, 2019)

Towards Excellence in Schools (MOE, 1987)

Education is a dynamic process in Singapore with a zest for continuous improvements, related to all matters of schooling. A report 'Towards Excellence in Schools' (MOE 1987) was released by the MOE in 1987. This report noted that the goal of excellence in education could only be achieved through better schools (Tan, 1987). Better schools meant world class curriculum and teachers.

In 1988, the Curriculum Development Division of the MOE set up a mathematics syllabus review committee to review and revise the mathematics syllabuses in use then. The goal of the committee was to study the adequacy of the syllabuses in meeting the needs of the students and to revise the syllabuses to reflect appropriate recent trends in mathematics education (Wong, 1991). It was during this review that the committee felt that besides elaborating the aims and objectives, a framework was necessary to describe the philosophy of the curriculum and articulate the primary focus of mathematics in schools. This framework, which has garnered international repute, is shown in Figure 1. It states mathematical problem solving as the primary focus and connects the 'product' conception of mathematics and its 'process' aspect linking both to the five factors that facilitate the development of mathematical problem solving (Wong & Lee 2010). The framework "presents a balanced, integrated vision that connects and describes five components: skills, concepts, processes, attitudes and metacognition" (Leinwand & Ginsburg 2007, p. 32). The framework has remained steadfast since 1990, through every subsequent revision of the school mathematics curriculum.

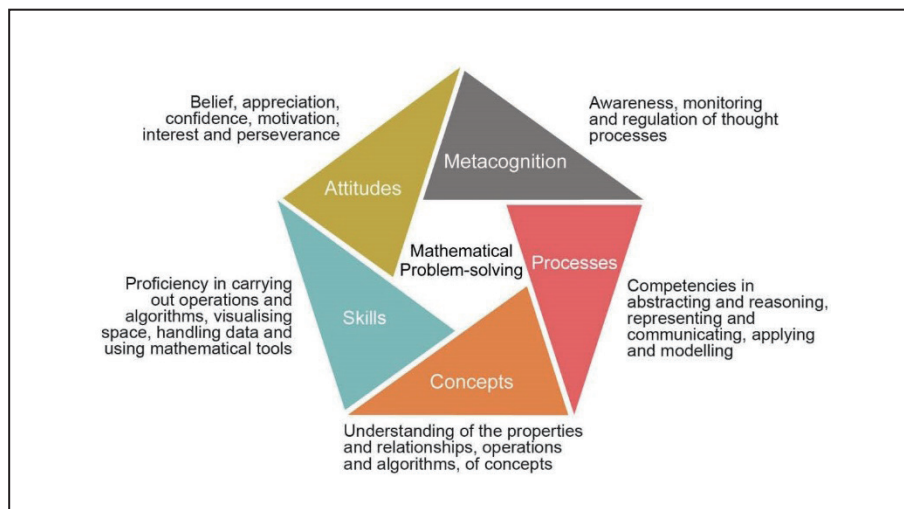


Figure 1. Framework of School Mathematics (MOE, 2018)

The curriculum has been reviewed periodically as of 1990. The review is guided by i) needs of economy and society; ii) global trends, so as to prepare our students for global citizenship, and iii) research carried out in schools specific to the teaching and learning of mathematics.

Two Key Education Policies That Impacted Mathematics Teacher Development

Thinking Schools, Learning Nation (Goh, 1997)

In 1997 at the opening of the Seventh International Conference on Thinking in Singapore the Prime Minister of Singapore signaled changes had to be made to the existing education system. These were necessary to prepare young Singaporeans for the new circumstances and new problems that they will face in the new millennium. He emphasized the need for our young to think for themselves, so that they can find their own solutions to whatever new problems they may encounter. He unveiled Singapore's vision for meeting this challenge encapsulated in four words: Thinking Schools, Learning Nation (TSLN) (Goh, 1997). Following the launch of the TSLN, MOE embarked on a systematic approach to induct teachers in lifelong learning (See Kaur & Wong (2017) for details), placing emphasis on continuous professional development so that schools keep abreast of advances in knowledge and learning at the national and international fronts.

Teach Less, learn More (TLLM) (MOE, 2005)

The Teach Less, Learn More (TLLM) initiative was introduced in 2005. It supplements the TSLN journey to improve the quality of interaction between teachers and learners. TLLM delves into the core of education - why we teach, what we teach and how we teach. It is about shifting the focus from "quantity" to "quality" in Singapore's education. It emphasizes "more quality" in terms of classroom interaction, opportunities for expression, the learning of life-long skills and the building of character through innovative and effective teaching approaches and strategies. It also emphasizes "less quantity" in terms of rote-learning, repetitive tests, and following prescribed answers and set formulae.

Centre for Research in Pedagogy and Practice (CRPP)

Since the setting up of the Centre for Research in Pedagogy and Practice (CRPP), in 2002, at the National Institute of Education in Singapore numerous research and development projects have had a significant impact on the professional development and practice of teachers. Due to the limit of space in this 4-page paper, I will only mention two past projects. The first is “Enhancing the Pedagogy of Mathematics Teachers (Teaching for Reasoning and Communication) (EPMT-RC) project (Kaur, 2009b; 2011) and the second is “A study of the enacted school mathematics curriculum” (Kaur et al., 2021). The deliverables of the first project, namely resources crafted by teachers (Kaur & Yeap, 2009a; 2009b, Yeap & Kaur, 2010) have contributed to several school-based professional activities, since 2010 till the present, that have had positive impact on classroom practice of many teachers in Singapore. The second project has made a significant contribution to literature on classroom practice of mathematics teachers in Singapore schools (see Kaur et al., 2019; Kaur & Leong, 2021; Kaur, 2022).

Research Agenda for a Digital Era

Bearing in mind, that:

1. Every revision of the school mathematics curriculum since 1990 has been a ‘nourishment’ in that the ‘baby’ has not been thrown out with the bath water, and
2. The quality of an Education System cannot exceed the quality of its teachers (Mourshed & Barber, 2007)

would direct a research agenda for school mathematics in a digital era that centers on the didactic relationships between **content**, **teacher**, and **learner**. Past research in technology-based instruction may call for upgrades and enhancements. This calls for research to inform teacher professional development in the digital era. Shifts from procedures to critical evaluations of machine produced solutions, self-regulated instruction, self-directed instruction and other learner centered modes may emerge relevant in the digital era. Research on all these aspects of student learning warrants research for teacher understanding and development.

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KOREAN MATHEMATICS EDUCATION AT A CROSSROADS: BALANCING TRADITION, INNOVATION, AND EQUITY

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This paper examines the evolution of mathematics education in South Korea through the lens of a system at a crossroads—one that must balance its long-standing tradition of academic excellence with emerging demands for innovation and equity. Drawing from historical trajectories, curriculum reforms, and the integration of AI and competency-based learning, the paper explores both the achievements and tensions inherent in Korea's current educational moment. The study underscores how balancing standardized rigor with dialogic pedagogy, high-stakes testing with inclusive practice, and centralized control with professional autonomy defines the path forward for Korean mathematics education.

Keywords: Korea, mathematics education, tradition vs. innovation, AI in education, educational equity, curriculum reform

Introduction: A System at a Crossroads

Korean mathematics education is globally recognized for its strong student performance in international assessments such as PISA and TIMSS. These outcomes reflect a historically centralized curriculum, a high societal regard for education, and rigorous teacher training. However, Korean education now faces a critical juncture—a crossroads where it must navigate the demands of 21st-century learning. As innovation and inclusivity emerge as central educational themes, questions arise about how Korea's traditional strengths can evolve to support new priorities. This paper critically examines the trajectory of mathematics education in Korea, focusing on the interplay of tradition, innovation, and equity.

Tradition and Strength: Centralization, Assessment, and Teacher Quality

Korea's post-war development of mathematics education was anchored in strong state control and a high-stakes examination culture. Since the establishment of the national curriculum in 1954, ten major revisions have taken place. The early reforms emphasized standardized achievement and academic excellence, underpinned by a Confucian educational ethic.

Teacher education and the multi-phase Teacher Employment Test (TET) ensured that secondary mathematics teachers possessed strong subject knowledge and pedagogical training (Kwon & Ju, 2012). This alignment between curriculum goals, public expectations, and teacher quality has undergirded Korea's international success.

In addition to structural coherence, the centralized model fostered a shared sense of purpose across regions and school levels. National textbooks, vetted by the government and implemented uniformly, contributed to consistent expectations. However, this centralization also stifled curricular creativity. Only in recent years have pilot programs allowed schools limited autonomy to adapt content based on student needs.

Moreover, teacher prestige remains high, but working conditions—such as large class sizes and administrative burdens—can diminish the pedagogical autonomy necessary for instructional innovation. Tensions between authority and autonomy persist within the teacher community, particularly among early-career educators.

To clarify Korea’s curriculum reform history, the following summary highlights key turning points:

Table 1. Major Phases of Korean Mathematics Curriculum Reform

Year	Reform Focus	Key Features
1954	Foundational curriculum	Standardized system introduction
1987	Efficiency & Restructuring	Streamlined content organization
1997	Constructivism Introduction	Emphasis on exploration & discovery
2009	Mathematical Processes	Reasoning and communication skills
2015	Competency-Based Learning	Alignment with global competencies
2022	AI & Personalization	Integration of digital textbooks

Innovation: Curriculum Shifts and Digital Transformation

Responding to global educational discourse, Korea introduced major reforms emphasizing competencies such as mathematical reasoning, communication, and modeling. The 2009 and 2015 curricula initiated this shift, culminating in the 2022 national curriculum which emphasizes AI integration and problem-based learning.

Middle school curriculum revisions in 2011 (Lew et al., 2012) aimed to reduce content overload and foster creativity and inquiry. Recently, the Ministry of Education initiated a plan to implement AI-powered digital textbooks by 2025, especially in mathematics (MoE, 2024; Park & Yoo, 2025). These tools promise personalization and data-informed teaching but raise concerns over excessive reliance on technology at the expense of human judgment.

Korea’s integration of AI in education is unfolding within a broader policy framework of ‘Smart Education,’ initiated over a decade ago. The forthcoming digital mathematics textbooks are part of a larger digital transformation initiative that includes personalized learning dashboards, AI-based tutoring, and data analytics platforms for educators. Pilot projects in select regions have shown early success in increasing student engagement and enhancing formative feedback loops. However, implementation gaps persist due to infrastructural disparities, digital literacy differences among teachers, and data privacy concerns.

A notable example is the Seoul Education Office’s AI-enhanced feedback system in middle schools, where students receive real-time suggestions during function graphing tasks. Preliminary data indicates improvements in students’ error detection and conceptual explanations.

Despite such innovations, change has been uneven across schools. Examination pressures and cultural inertia often limit meaningful pedagogical transformation (Pang, 2012, 2020).

Equity: Emerging Commitments and Lingering Challenges

Equity and inclusion are becoming more visible in Korean mathematics education policy and research, yet structural challenges remain. Standardized testing and the private tutoring culture disproportionately disadvantage students from marginalized backgrounds.

As Korea becomes more socioeconomically and culturally diverse, issues of linguistic accessibility, differentiated instruction, and mathematical identity are gaining attention. Performance-based and open-ended assessments are being piloted to shift focus from recall to reasoning (Hwang, 2021; Pang, 2020).

Another area of concern is the gendered dimension of mathematics education. While overall female participation in mathematics remains high, gaps in confidence and participation in higher-level mathematics tracks or careers persist. Policy efforts have begun to address this through outreach programs, mentorship initiatives, and textbook revisions aimed at diversifying representations.

Furthermore, the growing number of multicultural families and migrant students challenges the long-standing assumption of a linguistically and culturally homogeneous classroom. More research and policy experimentation are needed to ensure inclusive instruction that resonates with diverse student identities and experiences.

A recent initiative by the Gyeonggi Provincial Office of Education provides multilingual mathematics support in schools with large migrant populations. Early reports suggest improved participation and student confidence, though teacher training in cultural responsiveness remains limited.

The Balancing Act: Between Central Control and Professional Agency

Policy coherence has long been a strength of the Korean system, yet it may hinder local autonomy. Recent reforms have attempted to grant teachers more curriculum flexibility and promote inquiry-oriented practices (Sum & Kwon, 2020), but real change depends on school culture, leadership, and professional networks.

There is increasing recognition that sustainable reform depends on empowering teachers as co-constructors of curriculum rather than mere implementers. Policy consultations with exemplary teachers represent steps in this direction. Nonetheless, top-down expectations continue to dominate how new initiatives are interpreted and executed in practice.

Korea's professional learning ecosystem is rich, with subject associations, government-funded institutes, and school-based learning communities forming a dynamic support network. However, institutional constraints—such as rigid schedules, test-focused accountability, and limited teacher voice in policy design—remain obstacles to teacher-led innovation.

Professional learning communities and online teacher platforms offer promising avenues for bottom-up reform (Leong, Kwon, & Hino, 2024). Still, these networks must be sustained through time, recognition, and adequate resources.

Conclusion: Toward a More Balanced and Inclusive Future

Korea's mathematics education stands at a pivotal crossroads. The nation's historical strengths in structure, discipline, and academic rigor now face calls for greater innovation, equity, and adaptability. Future reforms must go beyond curriculum updates—they must transform classroom culture, empower teachers, and prioritize inclusive learning environments.

Korea's experience offers both lessons and cautions for other systems: lasting change demands not just policy mandates but a profound rethinking of what it means to teach—and learn—mathematics in an era of social transformation and technological disruption. To realize these aspirations, national policies must prioritize equitable resource allocation, professional autonomy, and collaboration with stakeholders beyond the education sector.

As East Asian countries collectively navigate similar tensions between standardization and innovation, Korea's evolving approach—rooted in cultural depth yet open to change—can serve as a generative model for regional dialogue, shared research, and collaborative policy design. Ultimately, the future of mathematics education in Korea will depend on its ability to balance tradition with transformation in ways that are both contextually grounded and globally engaged.

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THE CHANGING LANDSCAPE OF MATHEMATICS EDUCATION IN THE PHILIPPINES: REFLECTIONS ON THREE DECADES OF ACTIVE ENGAGEMENT IN ASIA AND BEYOND

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Mathematics education in the Philippines is probably at its most robust state now. This presentation highlights the activities and movements that have happened and are still happening in the field that have contributed to the changing landscape of mathematics education internally and outside of the country. It explores possible reasons and explanations for increased engagements by Filipino mathematics educators within the country, in Asia, and outside of Asia, individually or collectively, supported by government and work institutions or not. I end by highlighting what I believe are important challenges that we in Asia must face.

Keywords: changing landscape of mathematics education, mathematics education futures, Philippine mathematics education

Introduction

In this presentation, I share with all of you what I see as the changing landscape of mathematics education in my home country, the Philippines. This is a result of my own reading of and reflections on the last three decades of activities, events, developments, and setbacks that we have all had and continue to have and experience in my country.

Before I continue, however, let me declare my positionality in this exercise. I was one of few returning scholars in the Philippines in the early 1990s, after 5 years of graduate studies in the United States. Upon my return, I found myself working with academic and government institutions outside of my university on mathematics education matters at all levels — teacher training, graduate programs, policies affecting school curriculum and teacher preparation, scholarship programs, conferences and seminars, research, as well as governance. I helped found a professional organization dedicated to mathematics education. I also became active in the ICMI, being a country representative for 12 years who succeeded the long-time country representative Bienvenido Nebres and got elected as Member-at-Large in the ICMI Executive Committee in 2012 for four years. This is where I am coming from.

Factors That Influenced the Development of Mathematics Education

Mathematics education in the Philippines is perhaps at its most robust and exciting phase as a recognized academic and research field. Thanks to the efforts of the new players in the field — young mathematics education degree holders who have committed to serving the country and remained here. Indeed, there is a lot going on in relation to mathematics education concerns: increased enrollments in mathematics education graduate programs, active engagement in and of professional organizations in mathematics education and other related fields, focused research production and publication, and continuous engagement with the country's government institutions that oversee or relate to education. What may have contributed to these achievements and pursuits? I name a few:

(a) new personalities in the field; (b) regional and international activities, and (c) active engagement with and by government agencies.

New Personalities in Mathematics Education

I returned to the country at the time when there were large cohorts of graduate students enrolled in science and mathematics education programs who eventually finished their degrees. Credit must go to the Philippine Department of Science and Technology (DOST) who earmarked huge budgets for mathematics and science education development in the country through their capacity building programs as well as the Department of Education (DepEd) who opened graduate scholarships for mathematics teachers to pursue their master's degree in mathematics education (Department of Education, 2005). Younger mathematics educators who were offered local scholarships to pursue PhD degrees in the country quickly found faculty positions in key universities and colleges upon finishing, forming a critical mass of educators and researchers in the field. Being young and having gone through programs with new ideas for practice and research, many of us were slowly offered or elected to leadership positions in various endeavors. Now, we find ourselves working together and collaborating in activities and other engagements with the collective desire to improve mathematics learning among our pupils. This trend will hopefully continue as we see more and more scholars enrolling in these programs.

Regional and International Activities

A good number of regional and international activities easily found their way into the country and into the consciousness of local mathematics educators, prominent of which are conferences such as SEACME, EARCOME, ICME, and PME and large-scale assessments such as TIMSS, PISA, and SEA-PLM. Because of our desire to gain more knowledge in the field, benchmarking and networking initiatives increased. We would always seek opportunities to learn from our neighbors in Asia and from any country interested in working and collaborating with us on equal footing. The SEACME series began in 1978 and ended in 2002, when it was replaced by the EARCOME. The Philippines was host of SEACME 1 in 1978, and then again of SEACME 8 in 1999 (Vistro-Yu, 1999). We also hosted EARCOME 7 in 2015 (Vistro-Yu, 2015). Recently, the Philippines hosted the ICMI Study 27 Conference on "Mathematics education and the Socio-ecological." These international activities certainly contributed to our growth as mathematics educators individually and collectively as we learn the best teaching and research practices from all these.

Active Engagement with and by Key Government Institutions

Key government institutions continue to engage us mathematics educators in their activities and projects. In the last three decades, the country has had two revisions of the school curriculum and in each time DepEd consults the subject specialists. The latest of such consultations were held a year ago when a new curriculum was being prepared. Mathematics educators and mathematicians who have a special interest in mathematics education matters are periodically consulted at various stages in the development of the new curriculum. The Commission on Higher Education (CHED) is the government agency that oversees tertiary level education in the country. CHED also engages mathematics educators in its initiatives such as the preparation of college readiness standards and the development of the new course in the General Education Curriculum (Commission on Higher Education, 2013).

Key Shifts That Have Shaped Mathematics Education and Impacted Learning

Certainly, there had been important shifts that have shaped mathematics education in the Philip-

piners and impacted the learning by our pupils. These are the notable shifts that had concrete effects on our mathematics education practices, knowledge, and research: (a) Large-scale assessments: from local to global; (b) Leadership and key players: from mathematicians to mathematics educators; (c) Mathematics education on its own: from periphery to the center.

Large-scale Assessments: From Local to Global

Assessments in the Philippines are mostly confined to summative assessments in schools and entrance tests for special science schools or universities. Summative assessments such as quarter exams and the National Achievement Test (Department of Education, 2025) were meant to periodically gauge students' achievements in mathematics, science, and reading. But TIMSS 1995 and subsequently, TIMSS-R repeat changed the whole picture. When we had the chance to participate, we did, and Government made serious efforts in maintaining our participation in TIMSS. International benchmarking became a byword. The focus of assessments has now shifted from addressing pupils' deficiencies, for example, to preparation for participation in international assessments. We participated in TIMSS-R repeat (Brawner et al., 2000) and TIMSS 2003 (Cajilig et al., 2007; De Guzman et al., 2007), then again in TIMSS 2009. Recently, the DepEd has committed to participate more constantly in PISA and SEA-PLM. As a result, teachers are being trained to develop assessment items that approximate the level and type of items in international large-scale assessments. Certainly, this move has negatively affected our mathematics teachers as this adds to their responsibilities and takes them away from their primary duty of teaching good mathematics to our pupils.

Leadership and Key Players: From Mathematicians to Mathematics Educators

Another shift that we experienced was the gradual shift in leadership and the change of key players in mathematics education. Like in many countries, for a long time, mathematics education activities and initiatives were spearheaded by mathematicians. It was a natural thing for them since they are the specialists of mathematics. Education degree holders did not have much clout for the simple reason that they are not the subject specialists. More and more mathematics education degree holders took on leadership roles and became key individuals attending to mathematics teaching and learning matters, as mathematics education gradually grew and is now a flourishing academic and research field. A stronger group of mathematics educators has now emerged giving rise to shared goals and a common language with other educators and appreciated by mathematicians. It helped that a professional organization dedicated to mathematics education and mathematics teacher education was founded in the 1990s when mathematics education in Asia was also growing as a field. The Philippine Council of Mathematics Teacher Educators (MATHTED), Inc. founded in 1996 has been at the forefront, leading the country in promoting good quality mathematics teaching and mathematics education research.

Mathematics Education on its Own: From Periphery to the Center

There was a shift in perspective about mathematics education. Long known by and associated with 'teacher training' and clumped together with science education, it took long for mathematics educators to secure their own separate place in the academic fields or pursuits in the country. Mathematics educators were generally viewed as mere practitioners and not serious researchers. Moreover, for a long time, mathematics education was simply a part of Science Education. Hence, universities would offer programs with names like, "MS in Science Education major in Mathematics." Mathematics educators were therefore operating at the peripheries. But in the last three decades, mathematics educators have found themselves in the center of activities and are no longer confined to supporting roles. This shift proved to be beneficial because mathematics teachers can now easily talk with experts in the field who know of where and what they speak.

Challenges and Opportunities for Mathematics Education in Asia

Seeing how mathematics education has developed in the Philippines, I see several opportunities and challenges for mathematics education in Asia. The biggest opportunity I see is a productive regional collaboration in research. Atweh, Fan, & Vistro-Yu (2023) showcased the varied research that countries in Asia have been doing. The next step would be for countries to genuinely collaborate and share best practices in teaching and research. Certainly, countries in East Asia, Southeast Asia, and South Asia have much to offer to one another in all areas of research – curriculum, assessment, pedagogy, socio-cultural concerns, and technology among others. In addition, future research endeavors in our region have a venue for dissemination. The Springer book series, *Mathematics Education – An Asian Perspective* offers a rich opportunity for us in Asia to share our research and knowledge.

I see two challenges in Asia. The first revisits an international concern that came about in the early 1990s. The unpredictability of our future forces us to wonder if mathematics is indeed for all and if so, what kind of mathematics and for whom? Half of countries in East Asia, Southeast Asia, and South Asia are known to be high performing in international assessments while the other half are lagging. We are certainly challenged to rise above this disparity in mathematics achievement and focus more keenly on helping one another achieve quality mathematics learning in our schools, colleges, and universities.

The second challenge I see relates to knowledge production. OECD's *The Future of Education 2030 Project* asserts that as we face social, economic, and environmental challenges, students need to develop not subject-specific skills alone but also need to cultivate their curiosity, imagination, resilience, and self-imagination, among others (Taguma, 2018). Sharing this perspective, I now ask what kinds of mathematical knowledges we should be propagating and how these mathematical knowledges can contribute to Asian pupils' individual and collective well-being. I look forward to discussing these ideas with our colleagues in this conference.

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PROGRESS IN MATHEMATICS EDUCATION IN JAPAN: A 30-YEAR RETROSPECTIVE AND FUTURE OUTLOOK

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In this paper, I report some aspects of progress in mathematics education in Japan over the past 30 years. I first provide an overview of the revisions of the Course of Study and relevant events in these years and clarify the historical background that has shaped the revisions in mathematics. To focus on the shifts in curriculum, pedagogy, and assessment, I then describe features of the present mathematics Course of Study to promote students' competencies in thinking mathematically through mathematical activities. I also report on how the shifts have impacted students' learning by looking at several questionnaire items of the assessment. Lastly, based on what was described, I foresee the importance of continued efforts to develop students' competencies in thinking mathematically. Nevertheless, the perspectives need to be broadened to include eliciting and fostering the potential of each student's ways of thinking, which can contain refinement of mathematical activities applicable to diverse educational contexts and learners.

Keywords: mathematics education in Japan, the Course of Study, mathematical thinking, mathematical activities

Revisions of Course of Study Over Thirty Years

Japan's school curriculum is centrally designed by the Ministry of Education (Ministry of Education, Culture, Sports, Science, and Technology—Japan [MEXT]). The national curriculum, known as the Course of Study (COS), has been revised approximately every 10 years since World War II. Over the past 30 years, Japan revised its COS first in 1998. There was a reduction in curricular content and instructional hours as the school week was shortened from 6 to 5 days. The Programme for International Student Assessment (PISA), launched in 2000 with a focus on mathematics in 2003, revealed that despite high scores, Japanese students struggled to apply school-acquired knowledge to real-world problems (OECD, 2004). This finding significantly influenced subsequent revisions of the COS.

The 2008 COS marked a return to pre-1998 content while incorporating international trends, such as an enhanced focus on statistics. The National Assessment of Academic Ability (NAAA) by the National Institute for Educational Policy Research (NIER) began in 2007 and has annually evaluated students' proficiency in Japanese and mathematics at the end of the primary and lower secondary school. The 2017 COS signaled a shift from a knowledge-based to a competency-based curriculum. *Ikiru Chikara* (zest for living), previously promoted as a major educational goal, became the foundation for Japan's conceptualization of competencies. The 2017 COS was fully implemented in 2020. However, the COVID-19 pandemic disrupted education through school closures and accelerated the adoption of Information and Communication Technology (ICT), supported by the "GIGA School Program: Global and Innovation Gateway for All" – a national initiative to provide each student with a device and high-speed internet access to promote digital learning in schools. In 2021, the Central Education Council (CEC) proposed a vision for future Japanese school education, with a major proposal of integrated enhancement of individually optimized learning and collaborative learning (CEC, 2021). In September 2024, MEXT published a discussion paper by the Expert Panel on future curriculum directions, initiating deliberations for the next COS revision (MEXT, 2024a).

Under these revisions over 30 years, the goal of mathematics education has consistently emphasized the development of knowledge, skills, ways of thinking, and attitudes through engagement in mathematical activities. Several historical factors have supported this direction. First, since 1958, the COS has consistently emphasized mathematical thinking as the goal of mathematics education (Hino, 2007; Nagasaki, 1990). Second, scholars have accordingly made efforts to classify and name various forms of mathematical thinking, thereby promoting their dissemination within school practice. Instructional methods aimed at fostering mathematical thinking have also been developed through lesson studies, leading to a growing body of practical research (e.g., Isoda & Katagiri, 2012; Koto et al., 1992; Nakajima, 2015). Third, Shimada and colleagues, who developed an open-ended approach with the aim of nurturing higher-order mathematical thinking, presented a framework that encompassed both the processes of mathematical development and those involved in solving real-world problems (Becker & Shimada, 1997). This framework bears similarities to Freudenthal's concepts of horizontal and vertical mathematization (Van den Heuvel-Panhuizen & Drijvers, 2014). In Japan, mathematical thinking focuses primarily on processes within the mathematics world. The work by Shimada and his colleagues can be said to have broadened the scope to include thinking processes that connect mathematics with real-life contexts, which may have contributed to the conceptualization of mathematical activities in these years.

Present Emphasis on Curriculum, Pedagogy, and Assessment

In the 2017 COS, all subjects were structured based on three key pillars of competencies: “basic knowledge and skills,” “ability to think, make decisions, and express oneself,” and “motivation to learn and humanity.” Moreover, it states that students should nurture subject-specific perspectives and ways of thinking for every subject. These are regarded as the “discipline-based epistemology” rooted in the essence of academic disciplines and subjects (Shimizu et al., 2023, p.13). The overall goal of mathematics education is to develop students' competencies in thinking mathematically by enhancing their mathematical perspectives and ways of thinking through mathematical activities (MEXT, 2018).

I focus on the two terms that constitute the goal. One is “mathematical perspectives and ways of thinking,” meaning “interpreting phenomena by focusing on quantities, figures, and their relationships, reasoning logically based on evidence, and thinking in an integrated and developmental manner” (MEXT, 2018, p. 23). As a discipline-based epistemology, these support and guide all the three key pillars of competencies required for thinking mathematically. It also explicitly states that each student should actively apply these when approaching problem solving with a clear sense of purpose (p. 23). The other key term is “mathematical activities,” which is stated as, “the process of interpreting phenomena mathematically, identifying mathematics problems, and carrying out their resolution independently and/or collaboratively” (MEXT, 2018, p. 23). To make these processes explicit, two cyclic processes, problem-finding and problem-solving, are shown with an illustrative framework (MEXT, 2018, p. 8). One cyclic process involves understanding real-life and societal phenomena mathematically, expressing and processing them, solving problems, reflecting on the solution process, and considering the meaning of the results. This process is equivalent to what is referred to as mathematical modeling. The other process involves formulating new problems by examining mathematical phenomena in an integrative and developmental manner, solving them through mathematical processing, and reflecting on the process of forming and systematizing concepts. These two processes are considered forms of mathematical activity. Additionally, as a foundation for these, “activities for expressing and communicating mathematically,” which aim to enhance language use in various situations, are also regarded as one form. Each of these mathematical

activities is positioned within the curriculum at each grade level.

It is evident that these aspects inherited the emphasis on traditional mathematics education in Japan. Nevertheless, this is not simply a continuation of tradition. Regarding mathematical perspectives and ways of thinking, emphasis is placed on the fact that they are activated by the students themselves and develop alongside the growth of their competencies. In this respect, it differs somewhat from previous conceptions in which mathematical thinking was treated as something to be acquired. Problem-finding and problem-solving processes, presented as forms of mathematical activity, are consistently emphasized throughout primary and secondary education. However, when mathematical activities were first included in the goal statement in 1998, the terminology and characterization differed between primary and secondary education. The present continuity highlights the progress made over the past 30 years in conceptualizing mathematical activities.

In Japan, the problem-based approach, in which students work on rich open problems and compare and discuss their problem-solving attempts, is well known (Stigler & Hiebert, 1999; Takahashi, 2021). This approach is ongoing along with the promotion of mathematical activities in the lessons. Currently, in all subjects, improvement of the lesson from the perspective of “proactive, interactive, and authentic learning” (MEXT, 2018, p.3) is actively being pursued through lesson studies and other initiatives. In mathematics teaching, lesson improvement targets all key pillars of competencies in thinking mathematically through mathematical activities.

Regarding mathematics assessment, the NAAA has been conducted since 2007. The results, together with relevant information on lesson improvement, are available at <https://www.nier.go.jp/kaihat-su/zenkokugakuryoku.html>, and have been used in every school to improve mathematics teaching (Hino & Ginshima, 2019). The results further informed the 2017 COS revision by identifying areas of students’ difficulties as is seen in the inclusion of the new content, “simple proportion,” in grade 4 as the preliminary step toward “proportion” in grade 5. Presently, the problems included in the assessment booklet spread over different competencies, as the goals of mathematical content at each grade level have been updated based on the three key pillars.

How have these shifts impacted student learning? The results of comparative studies by the International Association for the Evaluation of Educational Achievement (IEA) have highlighted that Japan faces challenges in the affective domain. Concerning this, there has been gradual improvement over the past two decades. Regarding the question “*Do you enjoy studying mathematics?*” percentage of positive responses was 65% and 39% by the 4th and 8th grade students, respectively, in the Trends in International Mathematics and Science Study (TIMSS) in 2003. These were far behind the international average (79% and 65%, respectively). The percentages increased gradually, and in TIMSS 2023, they were 70% and 60% for 4th and 8th grade students, respectively (MEXT, 2024b).

The NAAA questionnaire includes items related to students’ interest and engagement in mathematics learning. One such item asks, “*Do you try to think about how what you learn in mathematics lesson can be applied in everyday life?*” The 32.9% of the primary school students in 2014 and 14.4% of the lower secondary school students in 2015 responded, “Strongly agree.” By 2024, these positive responses had risen to 42.4% and 23.4%, respectively. Another item asked, “*When you don’t understand how to solve a mathematics problem, do you try different approaches instead of giving up?*” The 42.9% of the primary school students in 2014 and 33.5% of the lower secondary school students in 2015 responded, “Strongly agree.” By 2024, these figures had risen to 47.1% and 37.2%, respectively (NIER, 2024). Although the data were based on students’ self-assessments and had limitations, gradual improvements were evident.

Toward Challenges and Opportunities for Mathematics Education in East Asia

Mathematics education in Japan has increasingly emphasized the development of competencies by placing the processes of problem finding and problem solving—both within mathematics and in connection with real-world contexts—at the center of students’ learning environments. I believe it is important to sustain this shift and continue to nurture competencies in thinking mathematically, though there is still room for refinement in the current approach to mathematical activities.

In mathematics education, issues of equity, diversity, and inclusion have come to the forefront over recent decades, with growing attention to how curricula, teaching practices, and underlying ideologies reflect and address these concerns (Abtahi & Planas, 2024; OECD, 2023). In Japan, there is an emphasis on balancing personalized and collaborative learning to respond to the needs of diverse students, thereby cultivating learners’ capacity for self-directed learning (CEC, 2021; Nasu, 2022). The perspectives of mathematical activities can be broadened to meet the needs of eliciting and fostering the potential of each student’s way of viewing and thinking mathematically, which can include refinements of mathematical activities applicable to diverse educational contexts and learners.

Similar efforts are being made to develop students’ competencies in East Asian jurisdictions based on their respective cultures and policies (e.g., Leong et al., 2024). At the same time, it is essential to consider how we might move beyond these boundaries. The continued exchange of ideas and collaborations are both challenges and possibilities for mathematics education in East Asia.

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THE DEVELOPMENT OF MATHEMATICS EDUCATION IN HONG KONG: LOOKING BACK AND LOOKING FORWARD

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Abstract: This paper aims to discuss the development of mathematics education in Hong Kong SAR, China, contributing to the evolution of mathematics education in East Asia. Hong Kong students has attained outstanding achievement in international comparative assessments in mathematics (such as, TIMSS, PISA). The factors supporting these achievements are many, whereas the dynamic development of mathematics education is one of the most important. Curriculum reforms and teacher professional development are interactive and essential factors contributing to dynamic developments in mathematics education. Under the canopy of the major milestone of the mathematics curriculum reforms the author argues that teacher professional developments via school-based projects are essential elements in mathematics education. Drawing upon lessons from the past, the author suggests two questions for the research agenda for the future development of mathematics education in the digital era: (1) How may mathematics curriculum support students' development of digital literacy? (2) What may be essential in the teacher professional development for the digital era?

Keywords: mathematics education, Hong Kong, mathematics curriculum reforms, teacher professional development

Introduction

This paper will contribute to the theme “The Evolution of Mathematics Education in Asia” via looking into the development of mathematics education in the past in Hong Kong SAR, China.

Hong Kong is an urban city situated on the southeastern coast of China with dense population in which the majority is Chinese. Like many other places in Asia, Hong Kong has a rich cultural heritage emphasizing education, discipline, an endorsement of Confucian values (such as, respect for teachers, diligence, education values) and examinations, in addition, a robust positive attitude toward mathematics. Mathematics is a mandatory subject in the school curriculum and is regarded as fundamental, basics essential for learning many new objects as well as a marker of intelligence. Hence, people with good mathematics usually have better opportunities in both job hunting and further study. Mathematics is regarded as a mandatory school subject and fundamental to further study and students' career choice. Hong Kong students has attained outstanding achievement in international comparative assessments in mathematics (such as, TIMSS, PISA). The factors supporting these achievements are many, whereas the dynamic development of mathematics education is one of the most important. In this paper, the author aims to look into the development of mathematics education in Hong Kong in the past from two aspects: curriculum and teacher professional development. Curriculum changes and teacher professional development are interactive, and both are essential factors contributing to the dynamic development of mathematics education.

Curriculum Development in Hong Kong

The development of mathematics education in Hong Kong is under the canopy of continuous curriculum reforms. Below are the major milestones of the development of mathematics curriculum in Hong Kong:

- 1950s–1970s, Curriculum establishment post-World War II: The emphasis in the very early curriculum was on mastery of skills and examinations in the early establishment. Later, under the Modern Math Movement (Siu and Wong, 2023), the mathematics curriculum undergoing revolutionary changes, settling with the co-existence of the “Old Math” and “New (Modern) Math” syllabi.
- 1980s, Publication of multiple mathematics syllabi by Curriculum Development Council: The focus in the curriculum was on understanding and problem-solving, catering for different student needs and abilities, e.g. Syllabus A for students targeting at university and A-level study, Syllabus B for students targeting for vocational training and less math-intensive pathway. (Education Bureau, Hong Kong SAR.)
- 1990s-2000s, The Target-Oriented Curriculum (TOC) reforms: Under the impact of the National Curriculum Reform in UK, Hong Kong undergone TOC reforms, the curriculum was divided into 4 key stages underlining progressions, the content remained very much the same except putting emphasis on targets and progressions. In fact, the major changes were the introduction of the social constructivism approaches in the pedagogies, promoting communication and group work, that has developed a major impact in primary mathematics classrooms. The TOC reforms experienced a long path with a lot of debates, negotiations and efforts between stakeholders (government officials, teachers and university educators), and went through the draft phase, the formal introduction phase of TOC principles in curriculum guidelines, the implementation phase of teacher training, classroom practices, and assessment reforms.
- 2000s, Introduction of New Senior Secondary Curriculum (2009): There were several major changes in the education system: from 3+2+2 years to 3+3 years of secondary education, from 3 years to 4 years for university education, and the abolishment of the certificate-level and A-level examination, replaced with the DSE examination at the end of secondary education.
- 2010s, Continued Updated Curriculum Changes in alignment with Global Trends for the 21st Century: Updated Curriculum emphasizing STEM and digital literacy, integration of technology and 21st-century skills, mathematical modelling, etc.

What are the lessons underpinning mathematics curriculum reforms?

Experiencing these mathematics curriculum reforms, some emerging factors underpinning mathematics curriculum reforms can be observed:

- Top-down government policy, this in turns has a significant impact on timeline and resources.
- Alignment with international movement and global trends, hence, active learning, collaborative problem-solving, and the integration of technology have become prevalent.
- Teacher professional development including knowledge, interpretation, pedagogical skills about the rationale, principles and implementation of the reforms.

Teacher Professional Development in Hong Kong

Teachers are the key stakeholders in mathematics curriculum, and they are the key persons orchestrating favorable learning environments for our students. Thus, teacher professional development is essential. The teacher professional development in Hong Kong can broadly be categorized into professional studies and school-based projects.

Professional studies

Mathematics teachers receive their initial training in university programs at the beginning of their career, later many inservice teachers continue to enhance their academic qualifications via university programmes for Master degrees. In addition, addressing various issues such as medium of instruction, catering for diversity, small class teaching, they may attend short government commissioned programmes specially designed for these educational issues in the context of enhancing mathematics teaching practice at classroom level.

School-based projects (e.g. Lesson study, BYOD, STEM)

Professional development for curriculum reforms always has constraints such as time and resources and the need for teachers to put in extra effort and time. Often teachers express concerns for effective outcomes in their workplace, consequently, school-based projects sometimes can be a feasible approach. To give a few examples, the growth of teachers' collective pedagogical content knowledge in lesson study (Mok and Park, 2022), realization of teacher agency in continuous school-based projects in STEAM context (Mok, et al., 2024), overcoming challenges and enhancing technology enriched math lessons during the pandemic (Tsui and Mok, 2024). In these examples, the concepts of teacher agency (Priestley, et al., 2015) and teacher-researchers (Huillet, 2020) provide perspectives to understand how top-down policy and bottom-up practices interact with each other in specific school-based projects resulting in teacher professional development in Hong Kong. (Mok, et al., 2024).

Looking Forward: Research Agenda for the Digital Era

Curriculum changes and teacher professional development are interactive and essential factors contributing to dynamic developments in mathematics education. Looking forward to new development in the digital era, and drawing upon lessons from the past, the author suggests two questions for the research agenda for the future development of mathematics education in the digital era:

- (1) How may mathematics curriculum support students' development of digital literacy?
- (2) What may be essential in the teacher professional development for the digital era?

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**The 9th ICMI-East Asia Regional Conference on
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Invited Lectures



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



Invited Lecture 1

**RECENT DISCOURSE IN RESEARCH ON THE COMPETENCE OF
MATHEMATICS TEACHERS**

Gabriele Kaiser, Xinrong Yang, Johannes König

**EAST ASIAN NARRATIVES OF MATHEMATICS IN AN ARTS-BASED
RESEARCH PROJECT**

Caroline Yoon

**TRANSFORMING MATHEMATICS EDUCATION PRACTICES THROUGH
INNOVATIVE PEDAGOGY AND DATA-DRIVEN PATHWAYS**

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**TEACHING MATHEMATICS THROUGH APPLICATIONS:
A CASE OF MATHEMATICS EDUCATION FOR NON-STEM
UNDERGRADUATE STUDENTS**

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RECENT DISCOURSE IN RESEARCH ON THE COMPETENCE OF MATHEMATICS TEACHERS

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Several different paradigmatic approaches have emerged in recent discourse on the theoretical foundations of the teaching profession. The competence-oriented approach to teachers' professionalism has strongly influenced the discourse through its links with extensive large-scale studies (among others, the Teacher Education and Development Study [TEDS-M Study]). The various theoretical frameworks developed within this discourse, based on Shulman's approach to pedagogical content knowledge, differentiate between various knowledge-related domains enriched by affective and volitional factors. The discourse has evolved from an initially dominant cognitive perspective, with a focus on teachers' knowledge, to the inclusion of a situated perspective mainly conceptualized as teachers' professional noticing. After describing seminal conceptualizations in the field, we discuss the results of empirical studies on teachers' competence, instructional quality, and students' learning gains. We use examples of theoretical and empirical results from the TEDS-Research program, among others, to illustrate the development of the aforementioned discourse.

Keywords: competence research, teaching profession, mathematics teachers, situated perspective, cognitive perspective, teachers' knowledge, teachers' professional noticing

Introduction

Since the late 1990s, a growing body of empirical research has focused on teacher education and teachers' professionalization; research on teacher education, teachers' professional development, and the foundational requirements for both has evolved into a rapidly expanding and highly productive field, largely driven by ongoing critiques of the limited effectiveness of teacher education in fostering adequate professionalization among teachers (König et al., 2024). Furthermore, this development has been significantly influenced by international comparative studies at the student level, such as the Program for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS), which have drawn attention to the centrality of teachers' competence in shaping students' achievements. Recent large-scale assessments of the professional competence of pre-service and in-service teachers have catalyzed numerous national and international studies to investigate the specific competencies required to teach various subjects across different educational levels (cf. Blömeke et al., 2014). Researchers have conducted cutting-edge research to explore not only teachers' competence but also its impact on students' learning outcomes and the instructional quality that teachers provide, the latter acting as a key mediating factor (see recent TIMSS-based studies, e.g., Nilsen & Gustafsson, 2016).

Over the past few decades, teaching quality has been widely recognized as a key factor influencing both instructional quality and students' learning outcomes (Baumert et al., 2010; Hill et al., 2005). Nevertheless, most empirical studies on mathematics education have adopted cognitive lenses, focusing on the relationships between professional knowledge—such as mathematics content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK)—classroom teaching behaviors, and students' achievements in mathematics (Charalambous, 2020). This cognitively oriented perspective has recently been contested by scholars such as Depaepe et al. (2020), who contended that the interplay among these constructs is “far from simple and straightforward” (p. 181).

In response, researchers have proposed incorporating more situation-specific elements into the conceptualization of teachers' competence, including mediating elements that influence the connection between teachers' cognitive knowledge and their instructional practices, drawing on the concept of teacher noticing (Blömeke et al., 2015; Kaiser et al., 2017). Currently, there is a broad consensus among educational researchers that professional competence encompasses both cognitive dimensions (e.g., subject matter knowledge [SMK]) and situation-specific dimensions, such as teacher noticing, which comprises perception, interpretation, and decision-making.

Furthermore, there has been a growing effort to refine the conceptualization of teachers' competence, develop more sophisticated measurement approaches, and identify effective strategies for supporting its development (e.g., Santagata et al., 2021). Grounded in this expanded perspective, empirical researchers have examined how various components of mathematics teachers' professional competence interact to influence instructional quality and students' mathematics learning and development.

Overall, in this article, we discuss the development of the current discourse on teachers' professional competence, its structure, and how it is connected to instructional quality and students' learning gains. Particular emphasis is placed on the conceptualization of competence and related measurement challenges in teacher education and teachers' professional development. In current empirical studies, teachers' competence is evaluated and externally validated according to its effect on instructional quality and students' achievements. The discourse described in this paper concerns mathematics teachers, since mathematics is critical in both school and university settings. Mathematics is a foundational subject that often functions as a gatekeeper at the school level and for access to higher education. Hence, the preparation of mathematics teachers and their ability to deliver high-quality instruction are of major relevance for societies, school education policies, and the broader context of higher education. However, the discourse concerns many generic factors that are applicable to teachers' professional development in general.

Theoretical Conceptualization of Teachers' Professional Competence and Cognitively Oriented Approaches

The concept of competence in educational research has its roots in various theoretical traditions. In recent years, the definition proposed by Weinert (2001) has had a significant impact on empirical teacher education studies—especially large-scale studies—and teachers' professional development. According to this definition, competence encompasses not only cognitive abilities but also the motivational, volitional, and social components necessary for solving problems, as well as the corresponding volition and capacity to implement appropriate solutions. Weinert's definition has particularly influenced empirical research focused on the modeling and measurement of competencies, where it has been taken up to conceptualize competencies as context-specific cognitive dispositions for performance that are functionally oriented toward addressing situational demands within spe-

cific domains. This conceptualization integrates key elements of psychological approaches to competence and is based on the educational discourse over recent decades concerning the professionalization of teachers; it is based on currently dominant generic models of professional competence that encompass both cognitive and affective–motivational dimensions (Baumert & Kunter, 2006). Multidimensional frameworks for describing profession-related dispositions can be traced back to the influential work of Shulman (1987), whose identification of distinct facets within teachers’ knowledge bases has had a lasting impact on scholarly discourse (Blömeke, 2017).

Shulman’s seminal approach to teachers’ knowledge

Shulman (1986) suggested that effective teaching requires more than just knowing facts; it necessitates a deep understanding of the subject’s structure and the ability to convey its essence to students, called subject matter knowledge (SMK). Shulman (1987) introduced a foundational framework for understanding teachers’ knowledge, encompassing theoretical categories for the knowledge base, by analyzing the specifics of knowledge categories for the teaching profession. He distinguished between the following categories of knowledge for teachers:

- content knowledge (CK)
- general pedagogical knowledge (GPK)—the broad principles and strategies of classroom management and organization that appear to transcend subject matter
- curriculum knowledge, including the materials and programs that serve as “tools of the trade” for teachers
- pedagogical content knowledge (PCK)—the special amalgam of content and pedagogy that is “uniquely the province of the teachers, their own special form of professional understanding.” (p. 8)
- knowledge of learners and their characteristics
- knowledge of educational contexts—the workings of groups or classrooms, the governance and financing of school districts, and the character of communities and cultures
- knowledge of educational ends, purposes, values, and their philosophical and historical groundings

Shulman (1987) emphasized that among these categories, “pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching” (p. 8). He described PCK as the “blending of content and pedagogy” and as the “category most likely to distinguish the understanding of the content specialist from that of the pedagogue” (p. 8). Overall, Shulman put three categories in the foreground, all related to the content of the subject—“content knowledge,” “pedagogical content knowledge,” and “curricular knowledge” (1986, p. 9). He described CK as “the amount and organization of knowledge per se in the mind of the teacher” and foregrounded understanding of the structures of a subject in contrast to pure knowledge of its underpinning facts and concepts.

Further seminal approaches to understanding teachers’ knowledge and competence

Although Shulman’s work was groundbreaking and constituted a milestone in the development of theorization regarding teachers’ professionalization, critiques have emphasized that the knowledge facets are not sufficiently well defined to support empirical research. In particular, Shulman’s unclear distinction between CK and PCK in connection with mathematical knowledge as absolute, unidimensional, and static prompted outspoken criticism (Meredith, 1995).

The critique of Shulman's work led to other conceptualizations of teachers' knowledge. Fennema and Franke (1992) emphasized that in Shulman's approach, the critical word *transform* neglects the complexity of interactions between teachers and students, which are situated in specific contexts. They proposed modifying Shulman's model by conceptualizing teachers' knowledge as interactive and dynamic and distinguishing between components of that knowledge: knowledge of the content of mathematics, knowledge of pedagogy, knowledge of students' cognitions, and teachers' beliefs.

Departing from the critique that the two knowledge facets, CK and PCK, were not clearly distinguished enough, the US American Project Mathematics Teaching and Learning to Teach (MTLT) based at the University of Michigan defined and distinguished between different knowledge facets that are functional for mathematics teaching and the mathematical and pedagogical work involved (Ball et al., 2008). The central construct developed by the MTLT Project was the construct of Mathematical Knowledge for Teaching (MKT) as a component of quality mathematics instruction defined as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p. 373).

MKT embraces the two knowledge facets subject matter knowledge (SMK) and the pedagogical content knowledge (PCK)—but differentiates between their various sub-facets. SMK includes both the mathematical knowledge that is common to individuals working in diverse professions and the specialized mathematical knowledge required for teaching. It contains new strands beyond Shulman's conceptualization—common content knowledge (CCK) and specialized content knowledge (SCK). CCK is, according to Hill et al. (2008), what Shulman probably meant by the construct SMK; that is, the content knowledge used in the work of teaching in the same way as it is used in many other professions or occupations that also use mathematics. SCK is an additional conceptualization that describes the mathematical knowledge that allows teachers to engage in particular teaching tasks, such as representing mathematical ideas or providing mathematical explanations. The third sub-facet, horizon content knowledge (HCK), is defined more as an awareness of the wider mathematical landscape in which the present experience and instruction are situated than as practical knowledge. The second facet, PCK, is more strongly aligned with Shulman's conceptualization. It encompasses knowledge of content and students (KCS) and concerns teachers' understanding of how students learn specific content. The second sub-facet, knowledge of content and curriculum (KCC), refers to the arrangement of mathematical topics within the curriculum and ways of using curriculum resources and materials. Knowledge of content and teaching (KCT) covers knowledge of both mathematics and teaching, such as the introduction of new concepts (Ball et al., 2008; Hill et al., 2008).

A central achievement of MTLT Project was the development of instruments to measure teachers' mathematical knowledge, which were transferred to other countries and cultures (Hill et al., 2007), allowing the relationship between teachers' knowledge and students' achievements to be measured (Hill et al., 2005). However, this MKT framework has a few significant weaknesses, such as not including teachers' beliefs, despite clear evidence from empirical research that teachers' beliefs about the nature of mathematics or the genesis of mathematical knowledge strongly influence their teaching (Schoenfeld, 2011). Another problem is the proximity of various sub-facets, such as the sub-facet SCK and the sub-facets KCT and KCS, which are difficult to distinguish, especially empirically.

The theoretical approach of the Knowledge Quartet, developed by research groups at the British University of Cambridge, arose from research into teachers' mathematical content knowledge. It includes Shulman's theoretical conceptualization but incorporates characteristics of the Fennema and Franke model by categorizing the teaching situations in which mathematical knowledge becomes apparent. The purpose of the research study was to develop an empirically based conceptual framework with a focus on the mathematics content of lessons—mathematics SMK and PCK. The focus

of the research was therefore on identifying ways in which teachers' MCK can be used in practical teaching based on the following categories: foundation, transformation, connection, and contingency. The first category, foundation, refers to the knowledge that teachers possess and teachers' understanding of and recourse to academic knowledge. The other three categories "focus on knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself" (Turner & Rowland, 2011, p. 200). The transformation category departs from Shulman's notion of content knowledge transforming into pedagogically powerful knowledge forms and mainly refers to the use of instructional materials, teacher demonstrations, and the choice of representations and examples. The connection category refers to the coherence of the planning or teaching displayed during a lesson or a series of lessons and concerns the formation of connections between procedures and concepts, decisions about sequencing, and so on. The last category, contingency, describes teachers' responses to classroom events that were not anticipated during the planning stage. These contingent actions are adaptive and adequate responses to children's ideas that involve teachers' deviations from their agendas.

Another important theoretical approach developed by the German Cognitively Activating Instruction (COACTIV) project was also based on Shulman's approach, conceptualizing teaching as a professional activity and knowledge as the core of professionalism (Kunter et al., 2013). Departing from the theoretical approach to professional competence defined by Weinert (2001), teachers' competence has been described in the COACTIV project as the personal capacity to cope with specific situational demands. The COACTIV theoretical framework distinguished between four aspects of competence—knowledge, beliefs, motivation, and self-regulation—which were further differentiated into facets. Concerning MCK, COACTIV focused on a profound mathematical understanding of the mathematics taught in schools. Kunter et al. (2013) described three dimensions of PCK: knowledge of the didactic and diagnostic potential of tasks and their cognitive demands; knowledge of students' cognitions (misconceptions, typical errors, and strategies); and knowledge of explanations and multiple representations.

From an international perspective, a central question concerns the extent to which the various dimensions of teachers' professional competence and knowledge can be empirically differentiated across cultural contexts. Revisiting Shulman's original conceptualization of PCK as a synthesis of content and pedagogical knowledge, An, Kulm, and Wu (2004) conducted a comparative study of mathematics teachers' PCK in China and the United States, focusing particularly on the teaching of fractions, ratios, and proportions. Their findings indicated that, unlike their US counterparts, Chinese teachers primarily developed their PCK through school-based in-service training facilitated by expert educators as well as through sustained professional development activities, such as collaborative lesson observations and reflective discussions (An, Kulm, & Wu, 2004; Paine & Ma, 1993). The aforementioned studies collectively underscore the challenges of reaching a universally accepted definition of MKT and the means by which it can be acquired. As noted by Pepin (1999), such divergences also reflect broader differences in the conceptualization of mathematics pedagogy, known in German as *Mathematikdidaktik*. The continental European tradition tends to be grounded in educational, philosophical, and theoretical inquiry, often incorporating normative frameworks of teaching and learning processes. Conversely, the notion of transforming academic knowledge into forms accessible to learners—a process referred to in German as *elementarization*—is scarcely addressed in Anglophone contexts. From the outset, researchers considering mathematics knowledge and teacher education in English-speaking countries (Kaiser, 2002) tended to adopt outcome-oriented approaches, relying heavily on empirical investigations aimed at identifying the key predictors of effective teaching and learning. As Westbury (2000) emphasized, the dominant orientation of the US curriculum tradition has been organizational, viewing schools as institutions in which teachers

are expected to function as agents of an optimally designed educational system.

Teacher Education and Development Study in Mathematics (TEDS-M)

In light of these cultural variations, a compelling question arises as to whether it is feasible to conceptualize teachers' professional knowledge of mathematics within the framework of a comparative international study. A pioneering initiative in this domain was the international comparative study TEDS-M, which was conducted in 2008 under the auspices of the International Association for the Evaluation of Educational Achievement (IEA) (Tatto et al., 2008). Involving a representative sample of approximately 23,000 participants across 17 countries, the study was conducted to examine how national policies and institutional structures shaped the outcomes of mathematics teacher education (Tatto et al., 2012). Anchored in Shulman's model of teachers' professional knowledge, the study operationalized two key dimensions—MCK and MPCK—as outcome indicators for assessing the effectiveness of teacher education programs.

Similar to the studies described previously, TEDS-M was based on the competence approach developed by Weinert (2001) and conceptualized teachers' professional competence as the specific ability to cope with the professional demands of teaching, strongly relating it to action-oriented approaches.

Departing from Shulman's (1987) theoretical approach, like most of the projects and theoretical approaches described previously, TEDS-M considered MCK and MPCK the essential cognitive components underlying teachers' performance in the classroom. In contrast to other conceptualizations, GPK as another cognitive ability and affective-motivational characteristics, such as personality traits and beliefs, were included.

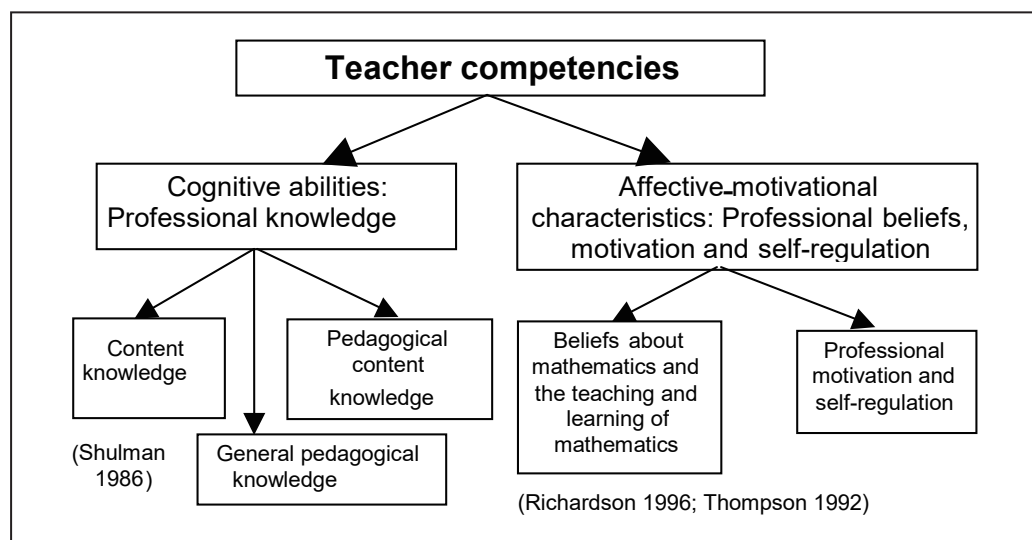


*Figure 1. Conceptual model of teachers' professional competencies
 (based on Döhrmann, Kaiser, & Blömeke, 2012, p. 327)*

As the first international large-scale study focused on teacher education, TEDS-M required extensive international collaboration to theoretically conceptualize MCK and MPCK and to develop valid competence assessments across various teacher education systems. This collaborative process resulted in definitions of MCK and MPCK that emphasized the actual tasks performed by teach-

ers rather than normative—often implicit—curricular prescriptions. Consequently, the conceptual framework for MCK was aligned with the content areas delineated in TIMSS 2007, for which international dialogue led to the identification of four key subdomains—numbers, algebra, geometry, and data—as foundational components of school mathematics. The MCK and MPCK assessments were structured accordingly, and the MPCK assessments distinguished between curricular knowledge, pre-active knowledge related to lesson planning, and interactive knowledge pertinent to classroom instruction. GPK was further disaggregated into subdomains, such as knowledge about teaching and learning, encompassing both classroom management and cognitive aspects, including theoretical and practical knowledge.

In summary, TEDS-M and related studies, such as COACTIV, drew upon Shulman's (1986) influential taxonomy of teachers' professional knowledge. In contrast, the Mathematical Knowledge for Teaching (MKT) framework developed under the umbrella of the MTLT project took a more focused approach, building primarily on the notion of PCK and identifying the specific domains essential for mathematics instruction. Other prominent studies investigating (prospective) teachers' professional knowledge have been based on alternative frameworks, such as the Teacher knowledge Developing in Context model, which emphasizes the situated nature of knowledge (Fennema & Franke, 1992), and the empirically grounded Knowledge Quartet framework, which supports lesson analysis with a particular focus on the mathematical and pedagogical content knowledge of future teachers (Turner & Rowland, 2011).

In the past, many empirical researchers investigating teacher education and teachers' professional competence have adopted cognitive perspectives. Their investigations typically followed analytical approaches aimed at defining and differentiating the specific facets of knowledge essential for effective teaching, with a strong emphasis on MCK. Conceptual models that portrayed the resources required for mathematics teaching proficiency as multi-dimensional have significantly shaped these frameworks, notably those proposed by Schoenfeld and Kilpatrick (2008), and further developed by Schoenfeld (2011), who characterized teaching as a knowledge-intensive endeavor involving both cognitive and affective–motivational components. Nevertheless, despite widespread recognition of the importance of teachers' knowledge—particularly the significance of Shulman's construct of PCK as a bridge between subject content and pedagogy and its subject-specific nature—empirical researchers have yet to achieve consensus on the constituent elements of this construct (Depaepe et al., 2013).

As Depaepe et al. (2013) noted, within the broader construct of PCK and its operationalization across empirical studies, two distinct perspectives on teachers' professional competence have emerged in recent decades. In addition to the aforementioned large-scale assessments employing differentiated models of teachers' knowledge or competence, a substantial body of research exists that does not explicitly disaggregate the components of PCK, even while focusing centrally on this construct. These studies have tended to be based on more holistic and situated approaches, or situation-specific conceptualizations of teachers' professional competencies, which will be further elaborated on in the following sections.

Situated Theoretical Conceptualizations of Teachers' Professional Competence

Cognitive and situated approaches to teachers' professional competence

In recent years, significant advances have been made in understanding teachers' competencies as personal traits (i.e., individual dispositions that remain relatively stable across different classroom situations; Kaiser et al., 2017), considering situational facets and social contexts that determine how competencies can be transformed into classroom performance as well. These developments correspond with current trends in subject-related discussions on education. Krainer and Llinares (2010) identified three prominent trends within the literature on prospective and practicing teachers, as well as teacher educators: an increased focus by researchers and educators on 1) the social dimensions of teacher education, 2) teachers' reflective practices, and 3) the structural conditions that shape teacher education. The first two trends, in particular, reflect a paradigmatic shift from viewing teacher education as an individual process to acknowledging its social embeddedness, informed by sociological and sociocultural theoretical frameworks.

This shift corresponds to the emergence of two distinct paradigms in the conceptualization of teachers' professional competence—cognitive and situated (Rowland & Ruthven, 2011). At a more granular level, the first two trends noted by Krainer and Llinares exemplify a situated perspective, emphasizing teachers' actions and the structure of their professional competence within specific instructional contexts. This represents a transition from the previously dominant cognitive paradigm, which emphasized individual knowledge components as central to teachers' professionalism. The cognitive view, prevailing in much of the past research, focuses primarily on knowledge dimensions and their organization into defined components that reflect individual traits. Examples of such studies were discussed earlier.

However, recent research initiatives, including follow-up studies of TEDS-M within the TEDS-Research Program (Kaiser, 2024), have extended the scope of inquiry by incorporating contextual and situated perspectives on teachers' competence into the analysis of teaching and teacher development. These studies have increasingly drawn on the concept of teacher noticing, which captures teachers' ability to attend to, interpret, and respond to classroom events. This approach underscores the multidimensional nature of teachers' competence, recognizing not only cognitive components related to subject matter, but also pedagogical and reflective elements concerning the overall teaching–learning process. Accordingly, the contextual nature of teaching has assumed a more central position in recent theoretical and empirical investigations.

These developments reveal a fundamental divergence within the ongoing discourse on teachers' professionalism, particularly regarding the conceptualization of PCK. Within the ongoing debate on teachers' professionalism, two principally different views on the conceptualization of PCK can be distinguished that point to general differences between the various studies; that is, “whether mathematical knowledge in teaching is located ‘in the head’ of the individual teacher or is somehow a social asset, meaningful only in the context of its applications” (Rowland & Ruthven, 2011, p. 3).

Adherents of the cognitive perspective:

... define – in line with Shulman – a limited number of components to be part of PCK and distinguish PCK from other categories of teachers' knowledge base, such as content knowledge and general pedagogical knowledge. By contrast, proponents of a situated perspective on PCK as knowing-to-act within a particular classroom context, typically acknowledge that the act of teaching is multi-dimensional in nature and that teachers' choices simultaneously reflect mathematical and pedagogical deliberations. (Depaepe et al. 2013, p. 22)

These paradigmatic differences in the conceptualization of PCK, according to Depaepe et al. (2013), influenced the way in which PCK is empirically investigated:

Advocates of a cognitive perspective on PCK believe that it can be measured independently from the classroom context in which it is used, most often through a test. They typically focus on gaps in individual teachers' PCK on how PCK is related to and distinguished from other categories of teacher knowledge base. ... Adherents of a situated perspective on PCK, on the contrary, typically assume that investigating PCK only makes sense within the context in which it is enacted. Therefore, they often rely on classroom observations (in some cases, supplemented with other data sources, such as interviews, lesson plans, logbooks) ... (p. 22)

Depaepe et al.'s (2013) analyses highlight the fundamental paradigmatic divergence among scholars concerning the conceptualization and assessment of teachers' professional knowledge and competence—including PCK—within the frameworks of cognitive and situated perspectives. In their synthesis, Depaepe et al. (2013) advocated taking an integrative approach that combines elements of both paradigms, acknowledging that each perspective possesses inherent limitations. For instance, the situated perspective may overlook the influence of sociocultural contexts, while the cognitive perspective may fail to account for the interrelatedness of various knowledge domains in teaching practice. Given that both approaches offer valuable theoretical and empirical contributions, Depaepe et al. (2013) emphasized the need to leverage their complementary strengths to deepen our understanding of the role of teachers' knowledge in shaping instructional quality and students' learning outcomes.

This divergence underscores the inherent complexity of defining and evaluating teachers' competence, particularly in light of the dynamic interconnection between individual cognitive resources and the socially embedded nature of instructional practice.

Competence as a continuum

The central question of how professional competence can be analyzed in a more performance-oriented way, including a situated perspective and situation-specific skills, has been further developed in Blömeke, Gustafsson, and Shavelson's (2015) seminal theoretical approach to competence as a continuum, which has greatly advanced the discourse.

Blömeke et al. (2015) proposed an integrative framework for conceptualizing teachers' competence that systematically incorporates the interplay of personal, situational, and social dimensions. Their model challenged earlier conceptual dichotomies that often presented competence as either a stable disposition or a context-dependent performance, thereby neglecting the dynamic interaction between these facets. Through a critical examination of existing conceptual debates and divergent definitions, the authors developed a comprehensive model that showed previously competing perspectives to be complementary. Within this framework, teachers' competencies are situated along a continuum that begins with relatively stable personal dispositions—such as professional knowledge and beliefs—that form the basis for situation-specific cognitive processes, including perception, interpretation, and decision-making. These cognitive processes ultimately manifest in observable teaching performance in classroom contexts.

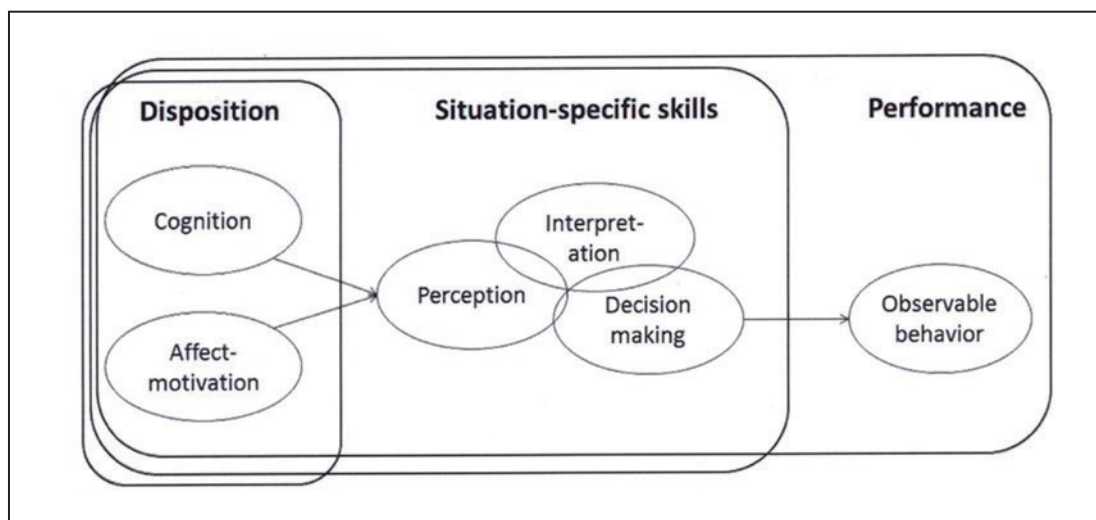


Figure 2. Model of teachers' professional competence from Blömeke et al. (2015, p. 7)

Teacher noticing as a central component of their competence

The situation-specific skills described as perception, interpretation, and decision-making are strongly connected to the discourse on teacher noticing, which has come to the forefront of discussions about teachers' professional competence and professional development in the last two decades, although no joint conceptualization or understanding of teacher noticing exists, as König et al. (2022) pointed out in their extensive literature review. In the literature, teacher noticing has been conceptualized in various ways. Scholars have often identified multiple facets or processes involved in noticing, such as identifying important and noteworthy classroom incidents, making connections between these incidents and the broader principles of teaching and learning, and reasoning about classroom interactions (van Es & Sherin, 2002). Other conceptualizations included in their definition deciding how to respond to (Jacobs, Lamb, & Philipp, 2010) or focus on the construct of attention to raise teachers' awareness of students and their understanding of the subject (Mason, 2002, 2009). Van Es and Sherin (2021) further developed the approach in more recent conceptualizations by incorporating the construct of shaping as the third noticing facet, which refers to the construction of interactions and contexts that provide access to additional information. Thomas et al. (2022) proposed enactment in an attempt to come even closer to classroom performance. In their literature review, **König et al. (2022) identified four major perspectives that have greatly shaped research on teacher noticing over the past two decades:**

- a cognitive–psychological perspective on teacher noticing that defines noticing as what teachers attend to and make sense of, focusing on the relevant mental processes (van Es & Sherin, 2002)
- a sociocultural perspective on teacher noticing that focuses on its social and situated nature and is often mentioned in conjunction with the term “professional vision” (Goodwin, 1994)
- a discipline-specific perspective on teacher noticing as practices for raising teachers' awareness (Mason, 2002)
- an expertise-related perspective on teacher noticing that focuses on the differences between experts and novices (Berliner, 2001).

Recently, as a further development, an ecological–embodied perspective on teacher noticing has been recognized, with noticing described as an embodied activity of exploring and engaging with the classroom environment (Scheiner, 2021). For a further overview, see Dindyal et al. (2021).

A widely quoted conceptualization of teachers’ professional noticing has been developed under the TEDS-Research Program to enrich the original cognitive perspective of the TEDS-M study and distinguish between the following three facets of teachers’ professional noticing (Kaiser et al., 2015, 2017):

- perceiving particular events in an instructional setting that correspond to attending to particular events in an instructional setting
- interpreting perceived activities in an instructional setting that correspond to making sense of events in an instructional setting
- decision-making, either as anticipating a response to students’ activities or as proposing alternative instructional strategies, which corresponds – although less straightforwardly than the first two facets – to acting.

Recent publications have highlighted the cultural nature of teacher noticing. For example, Louie (2018) explicitly stated that teacher noticing is “socially and culturally constructed” (p. 61). From a similar perspective, van Es et al. (2022) proposed a multidimensional framework of equity noticing that captures teachers’ attention to and sense-making of the local, sociocultural, and historical aspects of mathematics classrooms.

In connection with these developments, researchers have challenged the linear conceptualization of teachers’ professional competence, according to which teacher dispositions influence situation-specific skills and teaching behavior. For example, Santagata and Yeh (2016) proposed revising the model developed by Blömeke et al. (2015) and including a bidirectional relationship between teachers’ dispositional factors and their situation-specific skills, including their instructional practices.

Instructional Quality in the Discourse on Teachers’ Competence

Understanding and defining the features of instructional quality—or what constitutes good instruction—has long been the focus of educational research and policy, especially in research on teachers’ effectiveness. Studies of this type have been criticized for being undertheorized and focusing on easily measurable indicators. Notably initiated by international comparative studies, especially the TIMSS video study conducted in 1999 and national follow-ups, such as the German–Swiss Pythagoras study conducted from 2002–2003, conceptualizations of instructional quality have become diverse, sophisticated, and complex, and the approaches used to measure aspects of instructional quality have become a focus of educational research in their own right (Charalambous & Praetorius, 2018). Although most studies on instructional quality in mathematics education have not used formal frameworks (Bostic et al., 2021), such frameworks have been increasingly developed and used in the past decade. Relevant frameworks tend to differ according to whether they focus on generic (i.e., not specifically associated with teaching in a specific subject area) or content-specific (i.e., specifically associated with teaching and learning in a specific content area, such as mathematics) aspects of instruction, or, in some cases, both combined. Famous generic frameworks, such as the Classroom Assessment Scoring System (CLASS) developed by Pianta and Hamre (2009) and the three basic dimensions framework formulated by Klieme, Pauli, and Reusser (2009), had the advantage of facilitating investigations of instructional quality across subject areas. Especially regarding instructional research on mathematics education, the three basic dimensions framework has received considerable attention, and the following categories have been distinguished:

- efficient classroom management—optimal use of learning time in the classroom
- cognitive activation—offering the potential for challenging students to engage in complex thought processes
- constructive support—supportive learning environments with individualized and differentiated teaching offers.

However, for mathematics education, theoretical and empirical work, especially in the context of the TIMSS video study (Hiebert & Grouws, 2007), has suggested that important content-specific aspects of teaching and learning (e.g., the mathematical depth of argumentation or the adequacy of concept introductions) are not addressed in generic conceptualizations of instructional quality. Various subject-specific models have been developed, with the Mathematical Quality of Instruction (MQI; 2011) instrument being one of the most important. Within the framework MQI, various categories have been distinguished: connecting classroom practice to mathematical standards, working with students and mathematics, the richness of mathematics (e.g., representations and explanations), the occurrence of mathematical errors and imprecisions, and students' participation in classroom activities.

In addition to these two different theoretical approaches, hybrid conceptualizations have been developed in the past decade to integrate the subject-specific characteristics and generic aspects of mathematics teaching into various frameworks. A widely used framework in this respect is the Teaching for Robust Understanding (TRU) framework developed by Schoenfeld (2016), which distinguishes between the following categories: mathematics; cognitive demand; equitable access to content; agency, authority, and identity; and formative assessment.

Under the TEDS-Research Program a hybrid framework for instructional quality in mathematics education has been developed, departing from the three basic dimensions framework developed by Klieme et al. (2009) and introducing a fourth subject-specific characteristic of mathematics educational quality that includes two facets. The first facet—subject-related quality—focuses on dealing with students' mathematical errors, teachers' mathematical correctness, teachers' mathematical explanations, the mathematical depth of lessons, and support for mathematical competencies, such as problem-solving, modelling, or reasoning and proof. The second facet—teaching-related quality—covers using multiple representations, deliberate practice, appropriate mathematical examples, and the relevance of mathematics for students (Schlesinger et al., 2018).

The evaluation methods differed in the past across different frameworks that embraced video ratings or live ratings, which influenced the results to some extent, although the results were mainly comparable (Jentsch et al., 2024).

Overall, instructional quality has been identified as a central influential factor and mediator in impact models and recent approaches within the discourse on teachers' competence across the entire impact chain. This impact chain spans teachers' competencies, performance, and students' learning outcomes, and researchers have evaluated the impact of the various components along the impact chain. In the following sections, we describe the results of these empirical studies concerning the impact chain.

Results of Empirical Studies on Teachers' Competence, Instructional Quality, and Students' Learning Gains

Results regarding the structure of competence, the relationships between the competence facets, and their development

Empirical research in the past two decades has confirmed the multidimensional nature of teachers' professional competence (Blömeke, 2017), supporting the distinction in Shulman's (1987) framework between CK, PCK, and GPK. These domains have been theoretically and empirically separated across various studies despite the usage of differing assessment instruments (e.g., Blömeke et al., 2016; Krauss et al., 2008). This distinction facilitated a more nuanced analysis of teachers' knowledge, which is useful for evaluating teacher education programs and their influence on instructional quality and teachers' performance in classrooms.

Regarding the correlations between the specific facets of teachers' knowledge, PCK appeared to be closely related to both CK and GPK, supporting Shulman's "amalgam" hypothesis, while CK and GPK showed weaker associations. Among others, the COACTIV study found that CK and PCK were empirically distinct but strongly correlated with variations influenced by the type of education received (Baumert et al., 2010; Krauss et al., 2008). The findings of a cross-cultural study between Germany and Taiwan under the umbrella of the COACTIV study indicated that teachers' SMK is cross-culturally invariant and that CK and PCK represent two distinct but correlated dimensions (Kleickmann et al., 2015). As in the TEDS-M study for pre-service teachers, Taiwanese in-service teachers outperformed German teachers in CK and PCK (Blömeke et al., 2014; Kleickmann et al., 2015). Concerning GPK, a reverse finding was obtained in an enrichment of the TEDS-M study focusing on the US, Germany, and Taiwan, with German pre-service teachers showing higher or comparable GPK levels (König et al., 2011).

Studies conducted under the TEDS-Research Program, such as the TEDS-Follow-Up studies explored changes in teachers' knowledge during their development of teachers after teacher education into early career stages following teacher education. The findings showed that GPK improved substantially, CK declined slightly, and PCK remained stable. CK rankings remained consistent over time, but PCK rankings shifted, indicating developmental potential through practice (Blömeke et al., 2014). These findings highlight the significance of mentoring activities in teachers' early career stages and the need for longitudinal research.

Studies conducted under the TEDS-Research Program on the structure of noticing confirmed that noticing can be measured reliably and that it is possible to separate the three facets of noticing—perception, interpretation, and decision-making—empirically. In addition, the results highlighted that the facets of perception and interpretation are strongly connected, as well as the facets of interpretation and decision-making. These results confirm that noticing can be described as a knowledge-based construct (Bastian et al., 2022). Cross-cultural comparisons between Chinese and German teachers revealed cultural differences: Chinese teachers excelled in noticing from a mathematics pedagogical perspective, whereas German teachers showed strengths in noticing from a pedagogical perspective (Yang et al., 2019).

The video-based instruments developed by the TEDS-Research Program facilitated insights into the development of teacher noticing across groups of teachers with different levels of teaching experience, spanning pre-service teachers at the masters' level, early-career teachers, and experienced teachers. The studies carried out in Germany revealed significant progress in all aspects of teachers' professional noticing between pre-service teachers and early-career teachers, demonstrating the importance of teaching practice for the development of expertise in noticing. Furthermore, stagnation at least or even a small regression has been noted through a comparison of the noticing skills of early-career teachers and experienced teachers. After nearly five years of professional teaching practice, early-career teachers performed better than experienced teachers with 20 or more years of professional experience in all noticing facets although the decline of the noticing skills was only small, but statistically relevant (Bastian et al., 2022). In addition, early-career teachers performed

better on decision-making and mathematics-related items, as well as knowledge-based items related to recent mathematics pedagogy and general pedagogy topics, which emphasized the importance of knowledge and professional development activities for teachers' professional noticing (Bastian et al., 2024). These results contrast with those of other studies, including one from East Asia, in which a continuous, nearly linear increase in Chinese teachers' professional noticing was identified across three groups of pre- and in-service teachers, with significant differences between early-career and experienced teachers and only small differences between pre-service and early-career teachers (Yang et al., 2021b).

Finally, relationships between mathematics teachers' knowledge and their professional noticing have been examined in various studies—especially those conducted in Western countries—confirming strong relationships between teachers' knowledge and their noticing, especially concerning instructional decision-making (Jacobs et al., 2010; Kersting et al., 2010; Santagata & Yeh, 2014). In addition, these studies revealed that the development of noticing skills could be supported by a deep understanding of content-related knowledge (Sherin & van Es, 2009; van Es & Sherin, 2008). However, the results could not be confirmed in studies with Chinese teachers, in which only weak overall connections between mathematics teachers' knowledge and their noticing were identified. Among others, the noticing facets of interpretation and decision-making had a stronger correlation with teachers' knowledge than the facet perception (Yang et al., 2021a), which aligned with study results from Western countries described above (Yang & Kaiser, 2024). Apart from cross-cultural influences, the empirical results may have been influenced by the conceptualization of teachers' content knowledge or competence (Copur-Gencturk & Tolar, 2022).

Relationship between teachers' competence, instructional quality, and students' learning and development

The relationship between teachers' professional competence, instructional quality, and students' learning outcomes has been examined in several large-scale studies over the last two decades (Baumert et al., 2010; Blömeke et al., 2022; Hill & Chin, 2018; Hill et al., 2005; Kelcey et al., 2019; König et al., 2021).

One of the first analyses of this type was conducted by Hill et al. (2005) and revealed that teachers' subject knowledge was significantly related to students' achievements. Specialized content knowledge and skills used in teaching were of great importance in supporting students' mathematical learning gains. In further studies, Hill and Chin (2018) and Kelcey et al. (2019) examined the relationships between teachers' knowledge of students or mathematical knowledge, the instructional quality they implemented, and the achievements of the students they taught. Significant correlations among teachers' knowledge, instructional quality, and students' achievements were identified; however, the study by Hill and Chin (2018) reported considerable confluences of differences across students, classrooms, schools, and school districts. In addition, Hill and Chin (2018) confirmed teachers' accuracy in predicting students' performance on cognitive assessments as a specific sub-facet of teachers' knowledge related to instructional quality and students' learning outcomes.

The COACTIV study identified a strong positive effect of PCK on students' learning and development, mediated by the instructional quality of the teaching provided. Concerning instructional quality, the dimensions of cognitive activation and individual learning support played decisive roles. Despite its strong correlation with PCK, teachers' CK had lower predictive power for students' learning and development (Baumert et al., 2010).

Under the TEDS-Research Program König et al. (2021) examined the links between teachers' GPK, their situation-specific classroom management expertise as a facet of teachers' professional noticing

ing, their implemented instructional quality (measured using a classroom observation rating instrument developed by the TEDS-Research Program), and the mathematical learning and development of their students. The findings indicated that both GPK and classroom management expertise predicted instructional quality. In addition, direct statistical effects on students' mathematical learning gains were identified, whereas no indirect statistical effects via instructional quality could be identified. These results confirm the important role played by general pedagogical competence, consisting of knowledge and situation-specific skills, in the implementation of high-quality mathematics instruction.

In a further study associated with the TEDS-Research Program, Blömeke et al. (2022) investigated the relationship between the cognitive aspects of mathematics teachers' competence—MCK and MPCCK—and the situated aspects, such as perception, interpretation, decision-making, instructional quality (measured by classroom observations), and students' mathematics learning and development. In this study Blömeke et al. (2022) took another approach; instead of analyzing the direct effects of teachers' competence on students' learning and development, evaluated an impact chain with multiple mediation processes, aiming to open the so-called “black box” of these relationships by considering indirect effects based on mediation processes. The situated aspects of competence—teachers' professional noticing—and instructional quality mediated between the cognitive aspects of competence—teachers' knowledge—and students' mathematics learning and development. Teachers' knowledge had no direct effects on students' learning progress, emphasizing the importance of teachers' professional noticing skills as part of a situated perspective on teachers' competence. Overall, these findings confirm Blömeke et al.'s (2015) model of competence as a continuum, which assumes that these mediating processes, including noticing, are more proximal to classroom practice and students' learning gains than teachers' knowledge.

Overall, the study results by Blömeke et al. (2022) confirm, on the one hand, that teachers' professional noticing as a situation-specific skill is crucial for ensuring instructional quality. On the other hand, the findings corroborate that teachers' knowledge can be regarded as a precondition for professional noticing skills and, therefore, for students' learning and development. The analyses can be seen as opening up the “black box” and as a milestone in empirical research on teachers' competencies and their effects on students' learning. From a measurement perspective, the results confirm the need to take a situated perspective on teachers' competence, including more proximal measurement instruments, such as video-based noticing tests or classroom-based ratings of instructional quality, while retaining more distal instruments for the measurement of knowledge. Finally, the findings emphasize that teachers' competence must be conceptualized and empirically modeled as a multidimensional construct, and that its effects should be evaluated via complex effect chains.

In summary, empirical studies on teachers' professional competence and its relationship to students' learning and development conducted during the last two decades under the aegis of different large-scale studies have contributed substantially to the discourse on teachers' competence. They demonstrated that teachers' competence must be conceptualized as a multidimensional construct. The effects of teachers' competence on students' learning and development are only becoming manifest through complex impact chains involving professional knowledge, noticing skills, and instructional quality. This underscores the need to integrate cognitive and situated perspectives into the ongoing discourse on teachers' competence. However, this may not be sufficient; perspectives on the further development of the discourse may need to adopt the performance-oriented discourse on teachers' expertise, as delineated by Berliner (2001) or Stigler and Miller (2018), which may broaden our understanding of the teaching profession and lead to new perspectives on the development of the discourse.

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EAST ASIAN NARRATIVES OF MATHEMATICS IN AN ARTS-BASED RESEARCH PROJECT

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For the past several years, I have been conducting an Arts Based research project, writing a collection of stories and essays about the fictional family of a young girl, who starts school and begins her mathematics education. The stories are designed to challenge harmful narratives about mathematics that are prevalent in Western popular culture and in doing so, encourage mathematics-averse readers to reconnect with the subject. Although the stories are set in a predominantly English-speaking neighbourhood in Aotearoa New Zealand, they include and respond to East Asian cultural narratives of mathematics engagement and ability. I discuss how my stories relate to representations of mathematics in East Asian and Western popular culture, and to my identity as a Korean New Zealander. The collection is a tribute to elementary mathematics – its quirks and marvels and the maddening delight of learning about them – and is intended for parents, teachers, and those curious about how we learn.

Keywords: Narratives of Mathematics; Popular culture; Arts-based Research; East Asia; Elementary Mathematics

Introduction

For the past several years, I have been writing a collection of stories and essays about the fictional family of a young girl, Rosie, who starts school and begins her mathematics education (Yoon, 2025a). Her mother hates mathematics and has avoided it until now, whereas her father likes the subject and wants to help but can't always see what his daughter doesn't understand. Each story features Rosie and her parents encountering some difficulty with mathematics and is followed by an essay that draws on research to make sense of what has just happened. The collection is a tribute to elementary mathematics – its marvels and inconsistencies, and the maddening delight of learning about them – and is intended for parents, teachers, and others curious about how we learn.

My work aligns with an Arts-based research methodology, which uses artistic media, such as dance, film, photography, and writing, to “explore, understand and represent human action and experience” (Savin-Baden & Wimpenney, 2014, p. 1). One of the goals of Arts-based research is to create compelling and evocative expressions that unsettle the familiar and invite audiences to see the world differently (Barone & Eisner, 2011). I wanted to challenge the assumption that mathematical ability is fixed, innate, and even binary: that you are either good at it, or you are not. This assumption can cause people to avoid mathematics and can lead parents to label children as ‘bad at mathematics’ at the first sign of struggle. My goal was to show how many of our childhood difficulties with mathematics are not due to individual ability (or lack thereof), but to the social, cultural, political and historical forces that shape our experiences with the subject.

While working on this project, I found myself reflecting on some of the forces that shape my own writing, particularly through my cultural identity as a Korean-New Zealander. This kind of self-reflection is quite common for those engaged in Arts-based research, but it is underreported in academia, which tends to value the product of writing over the process. Insights into the writing process can be useful for those wishing to conduct similar creative work, so I decided to use the occasion of speaking at this conference to share some of the ways my cultural identity has influenced

my writing process. My lecture will be what Laurel Richardson calls a *writing story*, a “narrative about the context in which the writing is produced” (2000, p. 931) and will be largely autoethnographic. To ground my discussion of my writing process in an example, I will first share the opening story, *Ouch*, from my currently in progress collection. Afterwards, I will describe how my cultural identity shaped my decisions about characters and themes in this story, as well as other stories and essays in my collection. Cultural considerations are an important part of Arts based research, but the work can be confronting and confusing. I hope that by sharing some of my process, I can help others working through similar challenges of their own.

The story, *Ouch*, is set in Aotearoa New Zealand, the country of my birth, a Pacific island nation with an indigenous Māori population and a history of British colonisation. Recent immigration from the Pacific, Asia, and beyond, and a revival of Māori language and culture have enriched the country’s cultural landscape, but the dominant cultural influence is Western, the dominant language English. Rosie and her parents, Jenny and Tom, live in a suburban neighbourhood on the edge of Auckland, the country’s largest city and most ethnically diverse. Their home, 15G Tūi Drive, is one of many townhouses nestled between older, grander homes. Around the corner is a row of shops, and beyond that, the school Rosie will attend when she turns five. And on this day, the first clear day after a week of rain, Jenny has the windows down as she drives to pick up her daughter ...

Ouch

Jenny pulls into the carpark, switches off the engine, and grabs her old grey cardigan. She’s had it ever since she was pregnant with Rosie, a practical gift from a friend with children who knew what a first-time mother would need. With no zips or buttons that can scratch a baby, it hangs soft and loose in the front, much like a thin blanket with sleeves. Best of all, it’s machine washable, and these days Jenny threw it over her work clothes before picking up Rosie, who was invariably muddy, sandy, or covered in paint.

In true Spring fashion, it is cold in the shade and Jenny hugs the folds of her cardigan as she walks up the steps of the old villa. Once someone’s home, it is now a highly regarded centre for early childhood education with a long waiting list and fully qualified staff.

The corridor is lined with cubbyholes containing spare clothes, balled up socks, leaves, rocks and lumpy clay bowls made by little hands. Everything is set low to the ground, the chairs, the shelves, the coat hooks, the world shrunk down to tiny steps.

A few children are sitting at the craft tables in the main room, but Jenny heads towards the chime and chatter coming from the garden, almost bumping into a woman, who emerges from the office at the same time. It is Sunita, the senior teacher who oversees the daily running of the centre. She was the one who helped Rosie settle in on her first day all those years ago, soothing and reassuring the parents who needed it more than the child. Ever since then, a friendship has grown and when she sees Jenny arrive, her gentle eyes light up.

“She’s on the mat,” whispers Sunita, pointing beyond the craft tables to the far corner of the room.

Jenny grins. She loves watching Rosie play. It doesn’t happen often. Usually, one of the children will see her and cry out, “Rosie, your Mum’s here!” Then Rosie will dive under a table or crouch behind a chair, while her friends stand in front of her, giggling and blocking Jenny’s view.

Sunita and Jenny tiptoe past the craft tables.

“She’s had a good day. This morning, she was leading some of the children to make a scarecrow with the spare clothes. She’s so creative!”

Jenny chuckles. At four and a half, Rosie was one of the oldest and relished telling the younger ones what to do.

“But she had an accident just before you came.”

“Oh?” Jenny turns to her, alarmed.

“Just a small one. They were taking turns jumping off the treehut onto a mattress. Rosie banged her elbow running back up the stairs. I think she was trying to overtake Ethan.”

Jenny sighs. She has heard a lot about three-year old Ethan. A few weeks ago, he wore Rosie’s fancy shoes without asking, and one of the bows had snapped off. Rosie was inconsolable but Tom and Jenny had offered little sympathy. The shoes weren’t for playing, they were for special occasions only and Rosie had known that when she smuggled them in her backpack. But to Rosie, Ethan’s guilt was greater and whenever she saw the broken bow, she repeated the story of his crime. Jenny isn’t surprised she hurt herself trying to take his place just now.

“We gave her an icepack,” Sunita says softly as they approach the blue mat. “She wanted a plaster, but there was no blood, so I said no. I think that upset her more than her elbow.”

On the mat, Rosie and her friend Lani are kneeling next to a basket of *Duplo*, playing with Rosie’s shoes – not the infamous bow shoes, her usual everyday ones. Inside each shoe sits a plastic Duplo figurine. The girls are singing a song about sailing down the river as they push their shoe-boats under a Duplo bridge. It’s a precarious structure, tilted and skewed to one side and Jenny fights the urge to fix it. Sunita has taught her that. Don’t jump in too early, let the children find their own solutions.

On the other side of the mat is none other than Ethan, making ‘choo choo’ noises and kneeling over a Duplo train affixed with many figurines. Jenny is surprised to find him there with the girls, but he appears to be playing alone.

“We need more people,” Rosie announces to Lani, bouncing up and digging through the basket. She hasn’t seen her mother. Finding no plastic Duplo people, she notices Ethan’s creation for the first time, puts her hands on her hips and frowns.

“This will be interesting,” Jenny murmurs to Sunita.

“Ethan,” Rosie calls across the basket in her clear voice. “Can we please have some of your people?”

Ethan looks up, pulls his train closer.

“Ethan,” Rosie’s voice grows louder. “You’ve taken all the people. That’s not fair. You have to share.”

Sunita leans towards Jenny. “She’s such a negotiator.”

Jenny nods. “It’s in her blood. My dad’s a lawyer so she’s cursed!”

“It’s a great skill.”

Rosie storms over to Ethan’s side of the basket. “You can’t have all the people!” She’s shouting now, and the children at the craft table stop what they’re doing and stare.

“Look.” Rosie glares and points to the shoe boats on her and Lani’s side of the mat. “We have just two people. You have ...” She crouches and starts to count, pointing to the people on Ethan’s train one at a time. “One, two, three, four, five, six, seven, eight, nine, ten ...” Her recitation slows. “Eleven, twelve, thirteen, fourteen, sixteen...no, wait.”

Jenny, leaning forward, flinches.

Rosie counts again, this time starting from the other side of the train.

“One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, sixteen, eighty, ninety, twenty! You have twenty!”

As Rosie counts, the muscles in Jenny’s neck begin to tighten and suddenly she was back at her old primary school, standing on the mat, only this one was green, not blue. The class was looking at her and her teacher was smiling encouragingly. They’d been counting up in twos, each child jumping up and shouting the next number in the sequence when the teacher pointed to them. Two! Four! Six! Eight! Ten! Twelve! It was Jenny’s turn. She jumped up.

“Thirteen!”

The smile on her pretty teacher’s face disappeared.

“No, fifteen! Sixteen!” Jenny blurted out numbers as the class began to laugh.

“Don’t just guess. Think!” the teacher said, her voice sharp and irritated.

Jenny tried, but her mind was blank. She wanted to please her teacher, but hot tears were pricking her eyes and she couldn’t bear to look at her. The teacher sighed impatiently and told her to sit down.

And now, here stands Rosie, chest puffed, in charge of the world, unaware her counting is full of mistakes. Jenny feels a sharp pain in her hand and realizes her fingernails have been digging into her palm. She pulls the sleeves of her cardigan over her knuckles and turns to Sunita. “She gets that from me. I was hopeless at maths too.”

Sunita looks up quickly. “Oh, you mustn’t say that.”

“No, it’s true,” Jenny reassures her. “I don’t have a maths brain. I was always more verbal as a child. Her dad’s better with numbers. I’m just not a maths person.”

“Don’t say that,” Sunita repeats, quite firmly this time.

Jenny is taken aback. Sunita, who is usually so gentle and understanding, seems put off, almost angry. Maybe she thinks Jenny is exaggerating? She isn’t. All through school it was the one subject she’d failed.

Her mother would reassure her. “Don’t worry. Your father only got 13% for his School Certificate maths and he turned out fine.”

Her father would shrug and laugh that he had never seen his teacher so confused. “He genuinely thought I’d pass, despite all the evidence to the contrary. He didn’t believe an Asian could actually fail maths.” Jenny’s father could trace his Chinese heritage back to his great-grandfather, who immigrated to New Zealand in the early 1900s, well before his wife’s Irish ancestors.

Jenny learned to laugh off her maths failures, too. Once, while handing back a multiple-choice test, her high school maths teacher announced that a monkey would have done better than her. The class had laughed but Jenny had laughed the loudest and these days, she was comfortable enough to joke

about her mathematical shortcomings, even with strangers. It didn't bother her anymore, and she wasn't surprised to find Rosie was hopeless at it too. She smiles to show Sunita it's ok, but Sunita doesn't smile back.

"Jenny, you might think that about yourself. But you must never say that in front of Rosie." She draws breath to say more but is interrupted by the wails of a little boy in a sparkly cape, who has come in from the garden, eyes squeezed shut, holding his head. Anastasia, one of the younger teachers, steers him to where Sunita and Jenny are standing. Children gather, staring like drivers at a crash scene.

"He stood up underneath the monkey bars and bumped his head," Anastasia says.

"Oh dear." Sunita crouches to examine the injury. "Let's get you an ice pack." She turns to Anastasia. "Bring the others inside. Start story time." Then turning to Jenny, she says, "Please Jenny, remember. Never say that to Rosie."

Jenny watches in disbelief as Sunita takes the sobbing child's hand and moves towards the kitchen. She has been told off. But what for? Then she feels a tug on her cardigan and looks down.

"Mummy," Rosie is staring up at her, eyes sparkling. "What are you not supposed to tell me?"

Considerations of Culture in My Choice of Setting and Character

For mathematics educators working in English-speaking 'Western' countries, Jenny's speech about not being a "maths person" and lacking a "maths brain" will be all too familiar. We hear this speech from students, parents, friends and family, who offer it as a punchline, an explanation, a statement of fact. Yet the speech is extreme. It ascribes mathematical ability to an entire brain, an entire identity, implying it is something you are born with, something you either have or lack. Critics point out this speech is seldom heard in East Asian countries, where mathematics is considered a skill that can be trained (Mok, 2020) and success is attributed to effort and preparation (Leung, 2006), not some mythical 'maths brain.' Yet the speech persists, which suggests it serves a purpose. As an act of self-protection, it separates the speaker from the mathematics that has harmed them and warns listeners to do the same. But while it offers the speaker some protection, it can also cause damage, especially to young children who hear it from their parents and come to believe they too are destined for failure (Boaler, 2022). This is why I decided to feature this speech in the opening story of my collection.

As a story about a parent being challenged for her mathematics-averse speech, it isn't necessary for Jenny to have East Asian heritage. Yet, Asians are highly underrepresented in English-speaking media, and in creating my fictional world, I saw an opportunity, and indeed, a responsibility as an Asian myself, to identify two of my main characters (Jenny and Rosie) as Asian. For a long time, however, I didn't know how to do this. Initially, I wrote the story without mentioning Jenny's cultural heritage, then was disappointed whenever readers assumed she was White. Later, I wrote another story in which I identified Rosie's father, Tom, as Māori. Writing about Tom's Māori heritage was challenging. I was wary of cultural appropriation and misrepresentation, and tried to mitigate this by attending talks and workshops by Māori writers and reading what I could (e.g. Allen & Trinick, 2022; Meany et al., 2022; Parra & Trinick, 2018). But I found it even more challenging to write about Jenny's East Asian heritage, as this required me to visit difficult aspects of my own East Asian experience.

For the story to work, Jenny's speech needed to be believable, and as researchers have frequently pointed out, her speech is not typically East Asian. Jenny's mother's speech is also unusual for East Asian immigrant parents (Mok, 2020). This meant Jenny and her parents were unlikely to belong to the vast majority of East Asians in New Zealand who migrated to the country in the last few decades. There is, however, a longer history of Chinese immigration to New Zealand that stretches back to the 19th century, and I decided to locate Jenny's and her father's East Asian heritage there. I considered the possibility that both her parents might be Chinese but reasoned this would have marked Jenny as so visibly East Asian, her peers would have reacted to her mathematics-averse speech in ways that made her conscious of the anomaly, thereby tempering it. Jenny's frequent use of this Western speech would be more plausible if she also had European heritage, so I decided to identify her mother as Irish.

In writing Jenny's father as Chinese, I was conscious I was working against a common stereotype in the West, that 'Asians are good at mathematics.' This stereotype might seem like praise, but it is often accompanied by harmful assumptions, such as Asians are uncreative, unemotional, good at following rules but not natural leaders. This is visible in a scene from *Family Guy*, an animated sitcom that pokes fun at American culture. In one episode (Butler et al., 2006), Peter, the main character, a white American, sits at a desk, about to start a mathematics test. As others take out their calculators, Peter picks up a boy with stereotypically East Asian features, places him on his desk and prods him repeatedly with a pencil, saying, "C'mon, do math!" thereby reducing him to a machine. The stereotype is linked to the 'model minority' label, a term that arose in mainstream American media in the wake of the Civil Rights movement to describe the educational and financial success of Asians in the USA. It functions as a fragmenting hegemonic device, disrupting potential alliances among people of colour against a common oppressor by holding up the achievements of Asian Americans as an example of what can be achieved through compliance and hard work (Lee, 2009). The stereotype is particularly damaging for Asians in Western countries who are not good at mathematics. Along with bearing the psychological burden of not living up to expectations, they may be reluctant to ask for help and can even be excluded from targeted mathematics support because of their identity (I et al., 2024). Thus, writing about Jenny's Chinese New Zealander father who "only scored 13% in maths for his school certificate mathematics exam" is a form of activism, an example of counter-storytelling (Solórzano & Yosso, 2002).

Yet, even after knowing the backstory to Jenny's East Asian heritage and being determined to include it, it was still some time before I knew how to make it visible in *Ouch*. I considered slipping it in as a descriptive feature – almost an adjective – but this felt clumsy and tokenistic. At the same time, I didn't want to be too heavy-handed. While deliberating, I recalled a book by the New Zealand comedian, Raybon Kan, *An Asian at my Table* (2004). The title refers to his experience of being put on his school's quiz team, because it was thought his Asian appearance would intimidate the opposition:

School was easy. Everyone assumed I was good at maths. They gave me the accounting prize — I didn't even take the subject. "That boy... he looks right."

There was a TV quiz competition for schools. My school put me on the team. I said to the teacher: "Sir, you can't put me on the team, I don't know anything." He said: "Don't worry Raybon, just sit there and look Chinese." I said, "I'll try." We get to the studio and you can feel the fear coming from the other teams. "They've got someone Chinese on their team! They'll get all the fractions!" (R. Kan, personal communication, March 30, 2025)

The excerpt comes from material Kan wrote for a stand-up comedy set that was televised in New Zealand in the mid 1990s. This was a time of intense racism towards Asians, following the passing of the Immigration Act 1987, which stipulated immigrants should be selected on merit, not ethnicity or nationality. Earlier laws and practices were designed to maintain a “White New Zealand,” with a preference for immigrants from “traditional source countries” – namely Britain. After the change, the number of immigrants from Asia increased significantly, and media and politicians called it an “Asian Invasion,” riding on the energy of growing anti-Asian sentiment. It was a difficult time to be Asian in New Zealand, and during my formative teenage years, I learned to associate my appearance and identity with vulnerability. Few spoke out against the racism or even acknowledged it existed: many Asians couldn’t, due to language barriers. For those Asians who could, speaking out meant risking losing the little societal acceptance we were granted. I was often told, “I don’t think of you as Asian,” by those who meant it as a compliment. During this time, Kan challenged Anti-Asian racism through his comedy, by bringing it into the open and revealing it as absurd. Upon recalling the power of Kan’s humour, I decided to introduce Jenny’s Chinese heritage by having her father speak to the stereotype directly, rather than trying to integrate it into the story by more subtle techniques. This was done in the following excerpt:

Her father would shrug and laugh that he had never seen his teachers so confused. “He genuinely thought I’d pass, despite all the evidence to the contrary. He didn’t believe an Asian could actually fail maths.” Jenny’s father could trace his Chinese heritage back to his great-grandfather, who immigrated to New Zealand in the early 1900s, well before his wife’s Irish ancestors. (Yoon, 2025)

It feels somewhat anticlimactic that all my deliberations led to just four new sentences. But in the end, I decided this was all that was needed to make Jenny’s East Asian heritage explicit, and to poke fun at the stereotype of Asians being good at mathematics, while challenging their underrepresentation and misrepresentation in Western media.

Considerations of East Asian Representation in Story and Essay Themes

In my leisure time, I enjoy watching Korean television shows (with English subtitles), particularly K-dramas. And over the last few years, I have noticed that whenever mathematics features in these K-dramas, it tends to be portrayed quite differently to how it appears in Western popular culture. Portrayals of mathematics in Western popular culture typically promote a negative, avoidant view of the subject. For example, researchers (e.g. Darragh, 2018; Hall & Suurtmann, 2020; Epstein et al., 2010) have identified the following narratives in TV shows, movies, books, and games that are popular among children, teens and young adults in English-speaking Western countries:

Narrative 1: Those who are good at mathematics are socially incompetent nerds and geniuses

Narrative 2: Mathematics teachers are ridiculous, inept, deviant and insane

Narrative 3: Mathematics is irrelevant to daily life

Narrative 4: Mathematics is a secret code that is impenetrable and nonsensical

Narrative 5: Mathematics is terrible and should be avoided

In contrast, the K-dramas I watched featured ordinary people using mathematics comfortably in their daily life, and characters who were especially good at mathematics were conventionally attractive, successful, highly regarded members of society. I found similarly flattering portrayals in the literary fiction I read by Japanese authors (translated into English), such as Haruki Murakami (2009, 2010/2012) and Yoko Ogawa (2003/2008). As a mathematics educator, this delighted me.

But it also made me conscious of my cultural identity in a way that made me feel vulnerable. My stories and essays seek to challenge harmful narratives about mathematics that frequently appear in Western popular culture. I hadn't anticipated that in challenging these Western narratives, I might be aligning with narratives about mathematics in East Asian popular culture. I didn't want people to think I was promoting one culture over the other, and this compelled me to think more carefully about the themes in my stories and essays than I might have otherwise.

In the second half of this lecture, I will discuss how my themes relate to the five narratives of mathematics listed above that researchers have identified in Western popular culture (Darragh, 2018; Hall & Suurtmann, 2020; Epstein et al., 2010) as well as one additional narrative I noticed while consuming East Asian popular culture in my leisure time:

Narrative 6: Mathematics is a gatekeeper

For each of these narratives, I will describe their presence and/or absence in East Asian and Western popular culture. This part of my 'writing story' (Richardson, 2000) is not a formal study and should not be viewed as a robust comparison, due to the idiosyncratic way I selected East-Asian media. The narratives from Western popular culture come from studies of large samples of media that were systematically identified and rigorously analysed (see Table 1 for details).

Table 1. Studies analysing the portrayal of mathematics and mathematically able people in English-speaking Western popular culture media

Study and media type	Media identified by	Analysis of portrayals of mathematics
Darragh (2018) study of <i>Young Adult fiction books</i>	Students in school bookclubs (16) in USA nominated books for American Library Association's Teens' Top Ten Lists over 13 years	Author identified references to mathematics in 59 of 136 books, and thematically analysed all passages mentioning school mathematics or teachers
Epstein et al. (2010) and Mendick et al. (2007) study of <i>movies, TV shows, books</i>	Year 10 students (556) and second year undergraduate students (100) from the UK described up to two images of mathematics in popular culture	Authors analysed a selection of the images, and conducted focus group discussions and interviews with a selection of participants on their perceptions of the images
Hall & Suurtmann (2020) study of <i>TV shows, movies, websites, games, and reading materials</i>	Grade 4 and 8 students (156 in total) from Ontario, Canada listed their top three choices for each of the five media	Authors identified 42 titles in the students' top choices as containing mathematics and analysed each portrayal to identify themes

In contrast, I did not conduct a systematic investigation of K-dramas and Japanese literature but simply reflected on the portrayals of mathematics I noticed in four shows and two books (see Table 2). Three of these items were chosen without any expectation of finding mathematics in them (Coffee Prince; Weightlifting Fairy Kim Bok-Joo; 1Q84), two were chosen because they were advertised as relating to mathematics (Crash Course in Romance; Melancholia), and one was recommended to me by someone who said it reminded them of my stories (The Housekeeper and the Professor). Together, these six items form a biased and non-representative sample of East-Asian popular culture. Yet, they provide evidence of some of the ways mathematics is portrayed in East-Asian popular culture.

Table 2. *K-dramas and Japanese literary fiction in which I noticed the portrayal of mathematics and mathematically able people*

Title	Plot synopsis	Role of mathematics in plot
Coffee Prince (Lee, 2007) <i>K-drama</i>	Eun-Chan, a young tomboyish woman, dresses as a man to work in a coffee shop. Her male boss, Han-gyeol, begins to develop romantic feelings for her, not realising she is a woman.	Negligible
Weightlifting Fairy, Kim Bok-Joo (Oh & Kim, 2016-2017) <i>K-drama</i>	Bok-Joo, a champion weightlifter and university student, falls for a weight-loss doctor, Jae-Yi. Her childhood friend, Jun-Hyung, a swimming athlete at the same university, finds out about Bok-Joo's crush and teases her, then starts developing romantic feelings for her himself.	Negligible
Crash Course in Romance (Jo et al., 2023) <i>K-drama</i>	Chi-Yeol, a celebrity mathematics tutor at a top private academy, falls for Haeng-son, the owner of a store selling home-made food. Haeng-son's adopted daughter, Hae-Yi, is a highly successful high school student, who ends up attending Chi-Yeol's classes, and the three become involved in unravelling a murder mystery.	Central
Melancholia (Lee & Kim, 2021) <i>K-drama</i>	Yoon-Su, a talented and passionate mathematics teacher, befriends a quiet student, Seung-Yoo, a mathematical genius, in an elite, corrupt school. As the two uncover a cheating scandal in the school, they are accused of having an affair.	Central
1Q84 (Murakami, 2009, 2010/2012) <i>Japanese literary fiction</i>	Aomame, a professional hitwoman, is tasked with assassinating the leader of a religious cult. Her efforts lead her to reconnect with her childhood love, Tengo, who is a mathematics tutor at a cram school and unpublished writer.	Peripheral
The Housekeeper and the Professor (Ogawa, 2003/2008) <i>Japanese literary fiction</i>	A housekeeper with a 10-year old son is hired to work for a retired mathematics professor, who has only 80 minutes of short-term memory due to a brain injury. As the professor shares with them his love of numbers, their friendship develops.	Central

Several of the stories that follow *Ouch* in my collection contain themes that speak to the six narratives. These are: *Bananas*, *Pebbles*, *Skip*, and *Fish* (Yoon, 2025b). After comparing the portrayal of mathematics in Western and East Asian popular culture, I will describe the plots of my stories and show how their themes are positioned in relation to the six narratives.

Narrative 1: Those who are good at mathematics are socially incompetent nerds and geniuses

In studies of Western popular culture, when characters exhibit mathematical ability, it often functions as their core identity. Sometimes they are main characters, whose genius mathematical ability is the heart of the story, as in the movies *Good Will Hunting*, *Pi*, and *A Beautiful Mind*. These geniuses are often portrayed as eccentric, obsessive, and mentally ill, with their love for mathematics driving them insane (Epstein et al., 2010). But mathematically able characters are more frequently

cast in more minor, one-dimensional roles in which their mathematical ability comes at a heavy social price. They are nerds who struggle to relate to peers and rely on their textbooks for company (Hall & Suurtman, 2020). If they happen to use mathematically precise language, they are given quizzical looks or urged to stop using such big words and speak English instead (Hall & Suurtman, 2020). These stereotypes marginalise mathematical ability and reassure audiences that it is normal to be bad at mathematics.

In my consumption of East Asian popular culture, I noticed the stereotype of the mad genius mathematician in Ji Hyun-wook, a minor character in *Melancholia* (Lee & Kim, 2021), a former mathematics professor who is confined to a psychiatric care home after becoming unhealthily obsessed with a difficult mathematics problem. However, I found many more portrayals of mathematical ability that countered this stereotype. For example, one of the lead characters in *Melancholia*, Seung-yoo, is also a mathematical genius. Though he initially appears quiet and reserved, this is due to a traumatic event in his past, and as his mathematical abilities mature, he becomes confident and sociable. He is conventionally handsome, rides a motorcycle, wears a leather jacket, and is publicly lauded (including on national television) for his achievements in mathematics. His nemesis, Ye-rin, is also a top achiever in mathematics, as are several students in *Crash Course in Romance* (Jo et al, 2023), and all are portrayed as conventionally attractive and highly regarded by their peers for their mathematical success.

This analysis suggests a difference. In the Western media analysed by Epstein et al. (2010) and Hall and Suurtmann (2020), mathematical ability is associated with social awkwardness and insanity, whereas in the K-dramas I watched, it is frequently associated with attractiveness and societal admiration. In my stories, I have consciously avoided depictions of mathematical genius of any kind and have instead chosen to focus on ordinary people finding joy (and frustration) in ordinary elementary school mathematics. They focus on mathematical engagement, rather than ability, and therefore do not align with either of these portrayals.

Narrative 2: Mathematics teachers are ridiculous, inept, deviant and insane

In her analysis of Young Adult fiction written in English, Darragh (2018) found that school mathematics teachers were consistently portrayed in negative ways. They were objects of ridicule, or were described as sinister, incompetent, deviant or insane. Teachers of other subjects, such as English, music, art, drama and religion, were portrayed in more balanced ways, and sometimes played important roles as positive guides for the young protagonist. In contrast, mathematics teachers, were not cast as role models, but as negative characters with subhuman traits.

The depictions of mathematics teachers in two K-dramas are much more flattering. In *Crash Course in Romance* (Jo et al, 2023), the male lead, Chi-yeol, is a handsome mathematics tutor in a private academy, who has become very wealthy through his success. He is a celebrity, his face appears on billboards, and parents queue daily to secure the best seat for their children in his classes. As a teacher, he is passionate and flamboyant and punctuates his lessons with his signature high-kick move. In the end, he uses this high-kick to defeat a serial killer and rescue his love interest. Less flamboyant, but also handsome, is Tengo, a popular mathematics tutor in *IQ84* (Murakami, 2009, 2010/2012), who is successfully reunited with his childhood love. Another mathematics teacher in *Melancholia* (Lee & Kim, 2021), Ji Yoon-soo, is beautiful, intelligent, and highly principled. All three mathematics teachers are conventionally attractive, competent in their profession, successful in love, and highly regarded. In contrast to the portrayal of mathematics teachers as subhuman identified in Darragh's (2018) study, the mathematics teachers I noticed in East Asian media appeared almost superhuman.

While planning my stories, I was concerned by the overwhelmingly negative portrayals of mathematics teachers in Western media and resolved to counter this by writing about excellent teaching, modelled on real people and events. Rosie's teachers are competent, both in the mathematics they teach and in their human interactions, and they provide positive guidance to Rosie and her parents. For example, Sunita challenges Jenny's mathematics-averse speech in *Ouch*, and Rosie's Year 1 teacher, Ms. Patel, gives Jenny the tools and confidence she needs to help Rosie in *Pebbles*. Thus teachers in my stories are closer to the teachers in the East Asian popular culture examples than those identified in Darragh's study, although they are not intended to be superhuman.

Narrative 3: Mathematics is irrelevant to daily life

Hall and Suurtmann (2020) found mathematics is often portrayed in Western children's media as irrelevant to daily life, with characters who are good at mathematics shown as incompetent in more practical activities. Word problems are often parodied for being overly complex and contrived. In the popular American TV show NUMB3RS, which is aimed at adults, the tagline states "We all do math every day," but Esmonde (2013) points out that the show fails to feature ordinary people using mathematics in their daily lives. Rather, professional mathematicians are shown using advanced mathematics in their jobs as crime scene investigators, conveying the message that mathematics is a powerful tool for mathematical elites working on specialized tasks.

In contrast, while watching two K-dramas, I found two examples of ordinary mathematics being used by ordinary (that is, non-mathematician) characters in their daily lives. In the first episode of *Coffee Prince* (Jang et al., 2007), the lead character, Eun-Chan is shown sitting on the bus, calculating large sums using pen and paper to determine how much she needs to earn from her part-time jobs so she can pay her family's rent. Similarly, in a scene from episode 3 of *Weightlifting Fairy, Kim Bok-joo* (Yang et al., 2016), the title character is shown standing at an ATM, calculating large sums of money in her head as she tries to determine if she can afford to pay for a weight-loss programme offered by her romantic interest. Neither character is defined by their mathematical ability: Bok-Joo is a university weightlifting jock, whereas Eun-Chan is a part-time waitress, taekwondo instructor, seamstress, and takeaway delivery driver but both perform their calculations quickly and accurately. Their mathematical proficiency is not the central focus of the scene, there is no comment on how difficult the sums are, nor how well the characters perform them. Rather, it is presented as an ordinary life skill, like cooking or riding a bicycle, that anyone can perform – even weightlifters and delivery drivers. Mathematics is shown as useful: it helps the characters assess their needs and decide on their next steps.

The relevance of mathematics to daily life is a theme in my story, *Bananas*, which begins with Jenny asking a six-year-old Rosie a question. Jenny wants to show Tom how well Rosie's mathematics has progressed with her support and thinks she can solve it easily. But to her surprise, Rosie cannot answer. Jenny encourages Rosie to try, but Rosie continues to struggle. Eventually, the pressure is too much and Rosie screams, then walks out of the room leaving Jenny and Tom, silent and bewildered.

Rosie's reaction can be understood when we consider the question she was posed:

There are seven monkeys at the zoo. Each monkey has two bananas. How many bananas are there in total?

Earlier that day, Rosie had been reciting her doubles addition facts and answered "Fourteen!" whenever Jenny asked, "What is seven plus seven?" Consequently, Jenny expected Rosie would have no trouble with this question. She doesn't realise this question requires Rosie to know,

$$2 + 2 + 2 + 2 + 2 + 2 + 2 = 14,$$

not

$$7 + 7 = 14.$$

The accompanying essay explores how mathematics is used in ‘real world’ contexts, and how different contexts can lead to different mathematics. In this sense, it aligns with the portrayal of mathematics as useful in daily life, while also acknowledging challenges arising from the artifice of word problems.

Narrative 4: Mathematics is a secret code that is impenetrable and nonsensical

Epstein et al. (2007) found that mathematics is frequently portrayed in popular movies in the West as a secret code that only a few brilliant minds can crack, although the effort often drives them mad. This is typically shown through characters puzzling over mathematical symbols, shown in ways that suggest strangeness, impenetrability, and confusion. Similarly, mathematical symbols and language are often used in nonsensical ways in children’s media to convey a sense of confusion (Hall & Suurtmann, 2020). The message to the audience is that the mathematics is inherently difficult to understand.

In contrast, when Eun-Chan is on the bus, calculating the amount of money she needs in *Coffee Prince* (Jang et al., 2007), we see the numbers she writes on her notepad, and they also appear on the bus window above her head, as if written on a blackboard. Similarly, when Bok-Joo is working out how much money she needs to pay for the weight-loss programme in *Weightlifting Fairy Kim Bok-joo* (Yang et al., 2016), the calculations appear as subtitles on the screen. In both cases, the symbols are provided to make the mathematics intelligible and visible to the audience. The symbols are not used to suggest impenetrability but are used to enhance mathematical clarity.

The theme of symbols features in my story, *Pebbles*. Jenny becomes concerned when she realises Rosie (at 5 years old) doesn’t recognize the + and = symbols, even though her class has been working on addition. She shares her concern with Ms. Patel, Rosie’s teacher, and is shocked to learn that Ms. Patel hasn’t taught the symbols to the class. Jenny starts to doubt Ms. Patel’s integrity until Ms. Patel demonstrates how she is systematically building towards the formal symbols through physical and visual tools, such as tens frames and dot arrangements. The accompanying essay points out that although we associate mathematics with its symbols, most of them were developed relatively recently – from the 15th century onwards. Early mathematicians used physical objects, such as abaci, made from pebbles placed on trays of sand. Later, when mathematical sums came to be written down, they were typically written in full sentences. Symbols were invented or developed as a useful shorthand to make it quicker and easier to record mathematics. Thus, Ms. Patel’s move to introduce mathematical concepts using physical materials and visual tools before connecting them to conventional symbols reflects her desire for children to experience symbols as intelligible, rather than sources of confusion. This aligns more with the K-drama examples discussed that seek to make mathematics intelligible, than the examples from Western popular culture in which symbols are used as a synonym for impenetrability.

Narrative 5: Mathematics is terrible and should be avoided

Mathematics is often portrayed as a source of terror in Western popular culture. Darragh (2018) found Young Adult fiction used metaphors associated with hell, torture, and nightmares to describe school mathematics and characters rejoiced whenever they avoided it by cutting class. Similarly, the main character in the movie, *Pi*, identified in Epstein et al.’s (2010) study, is tortured by his obses-

sion with mathematics and takes a drill to his own head to destroy his mathematical abilities. Later, he is shown sitting on a park bench, unable to do mathematics, but seemingly happy and at peace. Whereas these characters go to drastic lengths to avoid the mathematics that torments them, the main characters in the K-drama *Melancholia* (Lee & Kim, 2021) are drawn to mathematics because of its difficulty. After spending all night working on a problem without solving it, Yoon-Soo explains that a mathematician's happiness is not determined by whether they solve a problem. Seung-Yoo asks why mathematicians keep solving problems. She replies, "When you are solving it, there's a tingling sensation, excitement, or anxiety that you feel. Whether or not you find the answer, it's the moment that you are most engrossed in it that enraptures you." (Kim & Kim, 2021). For Yoon-Soo, the joy of mathematics doesn't come from the success of solving a problem, but from being so absorbed that everything else seems to disappear.

One could argue that the characters in *Melancholia* only experienced this joy because of their exceptional talent. However, the unnamed housekeeper has a similar experience in *The Housekeeper and the Professor* (Ogawa, 2003/2012), despite having "hated mathematics so much [in school] that the mere sight of [a] textbook made [her] feel ill" (p. 23). When the Professor asks her son, Root, to find a more efficient way of finding the sum of consecutive numbers than adding them one by one, she is surprised to find herself intensely absorbed by the question and is euphoric when she makes a new connection that helps her understand the problem.

My story, *Skip*, also portrays the joy of being absorbed in making new mathematical connections. When Jenny tells Rosie's new Year 2 teacher that Rosie is having trouble with skip counting, she hands Jenny a hundreds chart and tells her it will help. Jenny is initially skeptical but later sees its value when she and Rosie discover patterns in the chart while skip-counting in twos. Over the week, Jenny and Rosie discover new patterns while skip-counting in fives, tens and threes. Hooked on the thrill, Jenny continues to explore skip-counting in other numbers, even missing a family movie to satisfy her curiosity. Like the Housekeeper, Jenny is surprised to find herself drawn into this mathematical exploration and wishes she'd had these experiences when she was in school. Thus, *Skip* shows that even those whose experiences of school mathematics have been nightmarish can go on to find it a joyful and absorbing activity.

Narrative 6: Mathematics is a gatekeeper

While watching K-dramas, I noticed frequent references to the gatekeeping function mathematics performs in Korean society. The school students in *Crash Course in Romance* (Jo et al, 2023) and *Melancholia* (Lee & Kim, 2021) aim for high marks in mathematics so they can get into top universities and elite programmes of study that will secure their future standing in society. Parents support them by enrolling them in expensive private tutoring and lobbying to secure their place in exclusive schools, clubs, and competitions. In some cases, this leads to questionable ethical practices, such as cheating and corruption. For example, in *Melancholia* (Lee & Kim, 2021) Seung-Yoo's rival Ye-rin's excellent results are largely achieved through cheating that is enabled by her parents, teachers, the school principal, and even government officials. As Seung-Yoo and Yoon-Soo investigate and expose the cheating scandal, Ye-rin's mental health unravels, and is only saved by confessing to the corruption.

The narrative of mathematics as a gatekeeper was not identified by any of the three studies of Western popular culture (Darragh, 2018; Hall & Suurtmann, 2020; Epstein et al., 2010). Initially, I found this absence curious as the gatekeeping role of mathematics in Western society is well known, with politicians frequently linking economic growth with success in STEM. But perhaps it is a matter of degree. In Korea and Japan, social stratification is much more strongly linked to educational perfor-

mance than it is in the West, with one's job, salary, and even marriage prospects influenced by the ranking of the university one attends. Consequently, when high school students in Korea take the 8-hour national university entrance exam, planes are banned from flying and businesses delay opening to facilitate optimal conditions. Although high stakes assessments exist in Western countries, they don't generate the intense pressure that surrounds student academic performance in East Asian countries, which has been blamed for high rates of teen depression and suicide. The deleterious effects of this pressure are so widely recognised that government initiatives have been made to counter it, such as reducing the intensity and frequency of testing, and regulating the private education industry.

In my story, *Fish*, I acknowledge the gatekeeping role mathematics plays in society. Jenny and Tom attend a parent information evening at Rosie's new primary school. It is their first night out alone for a long time, and the newness of it all makes it feel like a date. Rosie's teacher, Ms. Patel, gives an illuminating presentation about the difficulties children encounter in mathematics due to oddities in the English Language, and Jenny, a lover of words, finds herself intrigued and enchanted. But the date turns sour when Tom offends Jenny afterwards by assuming she won't help Rosie with mathematics. He points out that Jenny's avoidance of mathematics has stopped her from advancing her career, and says he wants things to be different for Rosie. This hurts Jenny's pride, and she declares she will help Rosie with mathematics, surprising them both. Although the gatekeeping role of mathematics influences her decision, I wanted to temper its pull by pointing to other compelling reasons for engaging with the subject. Ultimately, Jenny's resolve is formed while she recalls the delight she experienced in Ms. Patel's presentation, which sparked her imagination and piqued her curiosity. Perhaps it is idealistic to portray imagination and curiosity as more powerful pulls than social mobility, but in writing fiction, we create a world, and I wanted to heed the warning offered in the K-drama examples.

Discussion

Arts-based research seeks to illuminate the complexity and subtlety of human interactions, and considerations of culture are an important part of such work. In this lecture, I have shown how my cultural identity as a Korean-New Zealander has enabled me to analyse, respond to, and include narratives about mathematics from East Asian culture into my stories and essays. This kind of work can be difficult and can make one want to diminish one's cultural representation or avoid it altogether. I hope I have shown the value of leaning into this kind of work – the insights and opportunities for activism that might not otherwise have been available.

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MEMO

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TRANSFORMING MATHEMATICS EDUCATION PRACTICES THROUGH INNOVATIVE PEDAGOGY AND DATA-DRIVEN PATHWAYS

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The evolving landscape of mathematics education demands a reimagining of how mathematics is taught, learned, and assessed. This paper explores the transformation of mathematics education practices through the integration of innovative pedagogical strategies and data-driven decision-making. We examine how inquiry-based learning, collaborative problem-solving, and technology-enhanced instruction can foster deeper conceptual understanding and student engagement. Drawing on classroom examples and tasks, the paper presents a framework for aligning pedagogical innovation with data-informed instructional practices to enhance outcomes in diverse educational contexts. Thoughtfully integrated data-driven approaches not only improve student achievement but also cultivate critical thinking, resilience, and a lifelong appreciation for mathematics.

Keywords: Data-driven Problem Solving, Lifelong Learning, Mathematical Modeling, Computational Thinking

Introduction

The World Economic Forum's Future of Jobs Report¹⁾ 2025 highlights a transformative shift in the global labor market, driven by rapid technological innovation, evolving demographics, and economic uncertainty. These forces are not only reshaping job structures but are also amplifying the demand for specific skill sets, particularly those rooted in data science, analytical thinking, and mathematics (Seshaiyer, 2021). As the report projects a net growth of 78 million new jobs by 2030, the pressing challenge is not just job creation but ensuring the workforce is equipped to fill these roles. The fact that 63% of employers cite a skills gap as a major obstacle underscores the urgent need for data proficiency and analytical capabilities. These are no longer niche competencies—they are foundational to navigating and thriving in a digital economy.

Data literacy empowers individuals to extract insights from complex datasets, support evidence-based decision-making, and drive innovation across industries. Similarly, mathematical literacy and analytical thinking are essential for problem-solving in increasingly automated and data-driven workplaces. These competencies form the backbone of roles in AI, machine learning, business intelligence, and even non-technical fields that are being transformed by data. In essence, the future workforce must be fluent in the language of data and equipped with analytical tools to remain competitive and adaptable.

Investing in mathematics education and data literacy is not optional, it's imperative for economic resilience and personal career longevity. In this context, mathematics education emerges as a critical lever for preparing students to thrive in the evolving workforce. Traditional pedagogical approaches are increasingly inadequate in equipping learners with the necessary analytical, technological, and collaborative skills. Innovative teaching strategies—such as experiential learning, inquiry-based learning, learning by doing, challenge based learning and blended instruction—are gaining promi

1) <https://www.weforum.org/publications/the-future-of-jobs-report-2025/>

nence for their potential to foster deeper conceptual understanding and engagement. Simultaneously, the integration of data-driven pathways, including learning analytics and personalized learning platforms, offers opportunities to tailor instruction to individual needs and monitor progress in real time.

This paper explores how the convergence of innovative pedagogy and data-driven practices can transform mathematics education, aligning it with the competencies demanded by the future workforce. By examining current research and case studies, it aims to provide a comprehensive framework for educators and policymakers seeking to bridge the skills gap and prepare students for success in an increasingly complex and dynamic job market.

The paper is organized as follows. It begins by establishing a foundation for understanding data-driven problem solving and its significance in modern educational contexts. The next section explores the application of this approach within mathematics education, highlighting how data informs curriculum design, assessment, and learner support. This is followed by a discussion on instructional shifts and emerging frameworks in data literacy that enable educators to make informed, responsive decisions. Subsequent sections examine how data drives instructional practices, enhances technology-enabled learning, and fosters the development of durable skills such as problem-solving, collaboration, and adaptability. The paper concludes by discussing data literacy as a framework convergence, emphasizing its role in unifying pedagogical innovation with data-informed strategies to transform mathematics education for the future.

Understanding Data-driven Problem Solving

The Pre-K–12 GAISE I (Franklin et al., 2007) was a visionary document that championed the necessity of data literacy starting in the early school grades. It provided a framework of recommendations for developing students’ foundational skills in reasoning in three levels across the school years, described as levels A, B, and C. The three level descriptions helped to outline the NCTM (2000) recommendations for data analysis concepts that should be learned in the Pre-K-2, 3-5, 6-8, and 9-12 grade bands. The GAISE document outlines a framework that consists of four interrelated processes including, formulating questions, collecting data, analyzing data, and interpreting results. Data-driven problem solving goes beyond that as a multi-stage process that begins with identifying a problem and concludes with the communication of insights or predictions. Each stage builds on the previous one and often loops back to earlier steps as new insights emerge. The typical stages include:

- *Problem Formulation*: The cycle starts with defining a meaningful question rooted in a local or global context—one that can be explored using data. Setting clear goals, objectives, and hypotheses is essential to guide the rest of the process.
- *Data Acquisition*: With a well-defined question, the next step is to gather relevant data. This may come from sources such as databases, surveys, sensors, APIs, or existing datasets. It is crucial that the data collected is accurate, relevant, and reliable.
- *Data Processing*: As British mathematician Clive Humby famously said, “Data is the new oil”—it holds value only when refined. Raw data often requires significant preprocessing, including cleaning missing values, removing duplicates, standardizing formats, and restructuring for analysis. This step is vital for ensuring data quality and enhancing the reliability of analytical outcomes.
- *Data Visualization*: Visual representation of data through charts, graphs, maps, and dashboards

helps uncover trends, patterns, and relationships. Effective visualization simplifies complex data and makes it accessible to a broader audience.

- *Data Modeling and Analysis*: With processed and visualized data, the next step is to apply mathematical, statistical, or machine learning models. These models help extract insights, test hypotheses, and make informed decisions based on the data.
- *Data Prediction*: Predictive analytics involves using historical data and trained models to forecast future trends or outcomes. This step is central to decision-making in fields ranging from business to public health.
- *Data Communication*: The final step involves presenting findings in a clear and meaningful way to stakeholders. This may include reports, presentations, or interactive dashboards, designed to inform action and support decision-making.

Also, it may be noted that this data-driven process is rarely linear. Discoveries in one phase—such as unexpected patterns in visualization—may prompt a return to earlier stages for further refinement or additional data collection. Throughout the cycle, ethical considerations, data privacy, and security are paramount to ensure responsible data use and maintain public trust.

Data-driven Problem Solving and Mathematics Education

Data-driven problem solving is deeply interconnected with foundational concepts in mathematics education. These foundations—such as number sense, computational fluency, mathematical modeling, computational thinking, and proportional reasoning—are not just prerequisites but essential tools that enable students to navigate the data-driven problem solving with confidence and precision.

At the core of analyzing data is the ability to build number sense, whether interpreting averages, comparing quantities, or identifying anomalies. Strong number sense and fluency with basic operations help students estimate, calculate, and verify results efficiently, which is critical when validating data or spotting errors (Seshaiyer, Suh and Freeman, 2011; Suh and Seshaiyer, 2012; Suh and Seshaiyer, 2013; Seshaiyer and Freeman, 2012).

Next, the process of framing real-world problems in mathematical terms is central to mathematical modeling which enables students to translate complex phenomena into quantifiable variables, develop equations or functions, and test their solutions—mirroring the stages of data modeling and prediction in data science (Suh and Seshaiyer, 2016; Suh and Seshaiyer, 2019; Seshaiyer and Suh, 2019; Seshaiyer and Lenhart, 2020; Suh et al. 2023).

Along with mathematical modeling, another important skill is proportional reasoning that helps them to make sense of data visualizations, geometric thinking and statistical relationships, such as interpreting per capita figures, understanding ratios, comparing growth rates, percent changes, scaling data and normalized comparisons—essential for meaningful data interpretation (Smith et al., 2013; Suh and Seshaiyer, 2016; Seshaiyer and Suh, 2022).

Along with this data literacy progression, the next skill for students to gain is computational thinking that underpins the logical and structured problem-solving needed in data science, including breaking problems into parts, recognizing patterns, abstracting from computation and designing algorithms. This supports data processing and analysis, especially when coding, automating tasks, or designing data pipelines as well as encourages students to reason abstractly and work with variables—skills drawn directly from math (Wing, J.M., 2006; Weintrop et al., 2016).

In summary, data-driven problem solving is not a separate skill set from mathematics—it’s an extension and application of it. Students who are grounded in mathematics fundamentals are better equipped to a) Ask data-informed questions; b) Make sense of messy or incomplete information; c) Create models that reflect real-world scenarios, and d) Communicate evidence-based conclusions. Integrating these mathematical foundations into data science education ensures that students are not just consumers of data, but active and ethical participants in data-informed decision-making.

Next, we will present some specific classroom examples that integrate various components of data literacy into the curriculum and how that leads to instructional shifts.

Instructional Shifts and Frameworks in Data Literacy

Imagine starting a mathematics class with the following “Data Talk”. Shown in Figure 1 is French Engineer Charles Minard’s depiction of Napoleon’s 1812 campaign into Russia (Tufte, 2001; Tarle, 2018) where Napoleon takes a massive army to attack Russia in June of 1812 but realizes that he had to retreat after reaching Russia. What do you NOTICE and What do you WONDER?

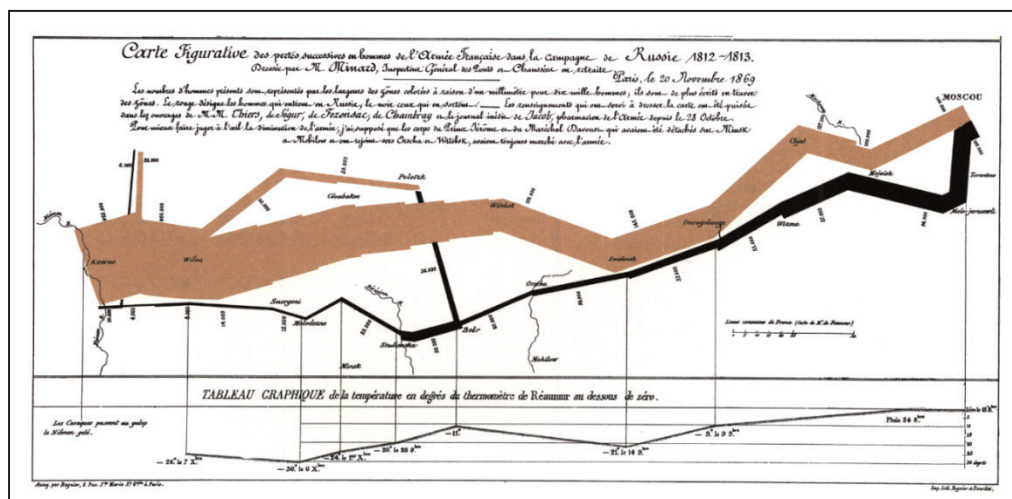


Figure 1. Data Visualization of Napoleon’s 1812 campaign to Russia

While this picture is evoked frequently as a major milestone in the history of cartography, the visualization is also notable for its representation in two dimensions of six types of data, both numerical and categorical data including geography, time frame, plummeting temperatures, the course and direction of the army’s movement, and the number of troops remaining. The size of the army was represented by the widths of the bands outward (gold) and returning (black) with one millimeter representing 10,000 men. Minard’s chart tells the dreadful story with painful clarity of Napoleon’s grand army that set out from Poland with a total of 422,000 men; only 100,000 reached Moscow; and only 10,000 returned (Hardy, 1992). The goal of such graphic representations is not necessarily to try to reveal new patterns, but to demonstrate that alternative representations give different perspectives and new insights into data. Incorporating multiple variables into a single graphic provides students with meaningful opportunities to make connections between visual data representations and historical narratives. When used in a mathematics classroom, such integrated activities not only enhance students’ data literacy but also support interdisciplinary learning. These approaches make data more accessible and relevant, enabling all students—regardless of background—to engage in deeper analysis and storytelling through numbers and images.

The ability to understand and interpret visualizations can be a highly engaging classroom activity and is a life skill in our data driven society. A classroom data talk is a low entry, high ceiling activity that engages all students in thinking critically about visualization. Every student can find a way to participate regardless of their background knowledge. Often, a data talk takes place at the beginning of a class to allow students to begin class with a universally accessible activity. A data talk may start with projecting a data visualization for the entire class to see (like Figure 1), having students look at a visualization on individual devices, or having a paper copy of the visualization for students.

The easiest way to begin a data talk is by asking students two questions: What do you NOTICE? What do you WONDER? Introducing students to a new concept this way by showing them a visualization and inviting them to report what they “notice” and what they “wonder” is a popular instructional strategy across subject areas (Rumack, A. M., & Huinker, D., 2019; Shumar, 2017). The classroom discourse from this can encourage students to engage in sensemaking and argumentation. Such an opening also helps to support students in feeling valued as their authentic selves as they are engaged in an interdisciplinary practice of interpreting and analyzing data. It also provides the opportunity for both the students and teachers to collaborate and construct the learning environment to make sense through discussion and learn more about the visualization.

Incorporating the power of data talk into a classroom as an important instructional shift that not only allows students to have the opportunity to analyze conventional and unconventional visualizations regardless of previous exposure but also helps them to discover and develop their creativity. Expanding this to include those daily moments of discovery through data talks, where clarity is reached and we say, “Oh! I get it!”, it outlines how an inquiry methodology like Notice and Wonder (Huinker, Bush, Graham, 2020) enhances creativity and allows students to view data from a place of wonder and excitement. Another important observation is how this data talk takes the classroom discussion beyond just one subject but integrates several subjects including history (timelines), geography (maps), science (temperature), technology (visualization engineering (drawing) and mathematics (graphs). By integrating notice and wonder strategies into data literacy instruction through data talks within instructional shifts and frameworks, teachers can foster a more engaging and meaningful learning experience for students, leading to deeper comprehension and critical thinking skills in data analysis.

Table 1. Instructional Shift in Data Literacy through Data Talks

Instructional Shift in Data Literacy	How to achieve it?
Data literacy instruction often involves traditional lectures and demonstrations, where students passively receive information. There needs to be a shift towards active learning methodologies where students actively engage with data through hands-on activities, discussions, and problem-solving. e.g. <i>Notice and Wonder</i> strategies through data talks can help promote active learning by prompting students to actively observe data, ask questions, and explore patterns.	<i>Moving from Passive to Active Learning</i>

Table 1 summarizes how using data talks through Notice and Wonder activities as instructional shifts can help student engagement from passive to active learning. Strategies such as data talks, encourage students to observe data closely, ask questions, and make connections, fostering a deeper understanding of mathematical and statistical concepts and data analysis. As teachers consistently use such powerful “notice” and “wonder” as drivers of instruction, it would stimulate all students and help to provide equitable access to disciplinary discourse and practices for multilingual learners (Delavan, M. G., & Matranga, A., 2020).

Data Drives Instructional Practices

An instructional framework in data literacy for teachers could include combining the understanding of data with other elements, including standards, disciplinary knowledge and practices, curricular knowledge, pedagogical content knowledge, and an understanding of how children learn. This holistic approach ensures that the process of transforming information into actionable knowledge is comprehensive and considers various factors that influence teaching and learning (Gummer and Mandinach, 2015). Specifically, five actions have been identified by different frameworks (Mandinach and Gummer, 2016; Kippers et al., 2018) for building a continuum of data literacy for teachers which include a) identify issue/opportunity and set goals; b) collect, manage, and organize high quality data; c) transform data into information; d) transform information into decision; and e) evaluate outcome. Accommodating such continuum throughout the grade levels is important to integrate real-world applications of data literacy, allowing students to apply their skills in various contexts. For example, emphasizing a strong foundation in understanding numbers and their relationships including counting, basic arithmetic, and recognizing patterns will help to introduce simple data concepts such as sorting, organizing objects based on attributes (e.g. color, size). This data literacy competency may be strengthened later in intermediate grades by introducing the students to basic data representation through charts, graphs and tables that they learn to interpret. In middle school, exploring proportional relationships and ratios where students get to analyze and interpret data in a complex manner can help to introduce computational thinking skills such as pattern recognition, problem decomposition, abstracting from computation and algorithmic thinking (Wing, 2006). This also involves engaging students to reason about several ideas or quantities simultaneously which will require them to think about situations in relative rather than absolute terms. Consider the following example. Showing a picture as in Figure 2 and asking students: “If one puppy grows from 5 kilograms to 8 kilograms and another puppy grows from 3 kg to 6 kg, which dog grew more?”

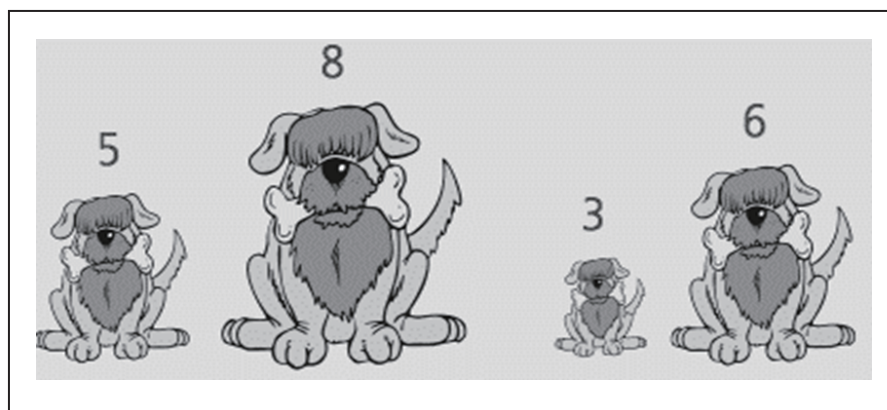


Figure 2. Illustration of the growth of two different dogs

Students in primary grades who often tend to think in absolute terms (or in additive terms) might answer that both puppies grew by the same amount. A middle school student may be thinking in relative terms might argue that the second puppy grew more since the previous weight was doubled, unlike the first dog who would have needed to be 10 kg to grow by the same relative amount. While both answers are viable, one could argue that it is the relative (multiplicative thinking) that is necessary for building data literacy as a learning progression as students move from primary to middle grades. Helping students bridge from additive thinking with data through such inquiry-based learning opportunities in primary grades to multiplicative thinking with data in middle school grades is complex but needs to start early. Learning progressions ground the learning goal of the curriculum where students are expected to make the transition from a lower level to a higher level. The lower levels are about intuitive ways of reasoning that students develop based on their earlier experiences in school and everyday life (e.g. absolute thinking), while the higher level often describes fundamental scientific reasoning (e.g. proportional reasoning).

Table 2. Instructional Shift in Data Literacy through inquiry-based learning

Instructional Shift in Data Literacy	How to achieve it?
Data literacy instruction should involve an inquiry based framework by emphasizing the importance of questioning in the exploration and understanding of data. <i>Inquiry-based learning</i> strategies align with this framework by encouraging students to generate their own questions about the data they encounter. These questions drive inquiry and exploration.	<i>Moving to a Inquiry-based approach for Data Literacy</i>

Such instructional shifts through inquiry-based learning (Table 2) aligned with learning progressions for this through data literacy form the backbone of mathematics and includes important and interconnected ideas such as multiplication, division, fractions, decimals, ratios, percentages and linear functions (Suh and Seshaiyer, 2015, 2016).

Data Drives Technology-enhanced Learning

Instructional shifts for data literacy through technology-enhanced learning involves evolving pedagogical approaches to leverage digital tools and visualizations effectively. These shifts aim to enhance students’ understanding, engagement, and proficiency in data analysis by incorporating technology and data visualization into interdisciplinary projects that require students to collect, analyze, and present data in meaningful contexts. Utilizing interactive platforms and simulations where students can manipulate data parameters, experiment with visualizations, and observe immediate results, fostering active learning and experimentation can help them to move beyond static learning materials to move towards dynamic digital environments that enable real-time data exploration and visualization. Consider the following data set from CODAP²⁾, that has information about 27 mammals with eight different data attributes including the Order, Lifespan, Height, Mass, Sleep, Speed, Habitat and Diet (See Table 3)

2) <https://codap.concord.org/>

Table 3. Dataset of 27 mammals with eight different attributes

in- dex	Mammal	Order	LifeSpan (years)	Height (meters)	Mass (kg)	Sleep (hours)	Speed (km/h)	Habitat	Diet
1	African Elephant	Proboscidae	70	4	6400	3	40	land	plants
2	Asian Elephant	Proboscidae	70	3	5000	4	40	land	plants
3	Big Brown Bat	Chiroptera	19	0.1	0.02	20	40	land	meat
4	Bottlenose Dolphin	Cetacea	25	3.5	635	5	37	water	meat
5	Cheetah	Carnivora	14	1.5	50	12	110	land	meat
6	Chimpanzee	Primate	40	1.5	68	10		land	both
7	Domestic Cat	Carnivora	16	0.8	4.5	12	50	land	meat
8	Donkey	Perissodactyla	40	1.2	187	3	50	land	plants
9	Giraffe	Artiodactyla	25	5	1100	2	50	land	plants
10	Cray Wolf	Carnivora	16	1.6	80	13	64	land	meat
11	Crey Seal	Pinnipedia	30	2.1	275	6	19	both	meat
12	Cround Squirrel	Rodentia	9	0.3	0.1	15	19	land	both
13	Horse	Perissodactyla	25	1.5	521	3	69	land	plants
14	House Mouse	Rodentia	3	0.1	0.03	12	13	land	both
15	Human	Primate	80	1.9	80	8	45	land	both
16	Jaguar	Carnivora	20	1.8	115	11	60	land	meat
17	Killer Whale	Cetacea	50	6.5	4000		48	water	meat
18	Lion	Carnivora	15	2.5	250	20	80	land	meat
19	N. American Opossum	Didelphimorphia	5	0.5	5	19		land	both
20	Nine-Banded Armadillo	Xenarthra	10	0.6	7	17	1	land	meat
21	Owl Monkey	Primate	12	0.4	1	17		land	both
22	Patas Monkey	Primate	20	0.9	13		55	land	both
23	Pig	Artiodactyla	10	1	192	8	18	land	both
24	Pronghorn Antelope	Artiodactyla	10	0.9	70		98	land	plants
25	Rabbit	Lagomorpha	5	0.5	3	11	56	land	plants
26	Red Fox	Carnivora	7	0.8	5	10	48	land	both
27	Spotted Hyena	Carnivora	25	0.9	70	18	64	land	meat

Presenting such an accessible dataset and asking students to make comparisons to learn from the data or even make scientific hypothesis on relationships is a great way to engage student learning through technology. For instance, students may compare two of the attributes such as Lifespan and Diet between the 27 mammals to discover patterns in the data or they could look at relationship patterns between Lifespan and Height (See Figure 3). As teachers engage students in such technological enhanced activities, it is important to understand that technology should not be used just for the sake of using if they do not understand the particular pedagogical role that software (such as CODAP) can serve (Franklin and Garfield, 2006). Students in this activity also had the opportunity to learn about the scientific method which is a structured, logical process used to investigate questions, solve problems, and understand natural phenomena. This framework goes very well with the mathematical modeling process (Seshaiyer and Lenhart, 2020). As illustrated in figure 3, the students were asked to “ask a question” which lead to the creation of a hypothesis. An example could be “*If a mammal is tall, then the mammal lives longer*” or “*A majority of the mammals eat meat*”. The students were then able to analyze the data, draw a conclusion and communicate their results. Figure 3 shows how students were able to create graphs using categorical or numerical approaches and the teacher used this instructional shift to introduce notion of linear regression and correlation.

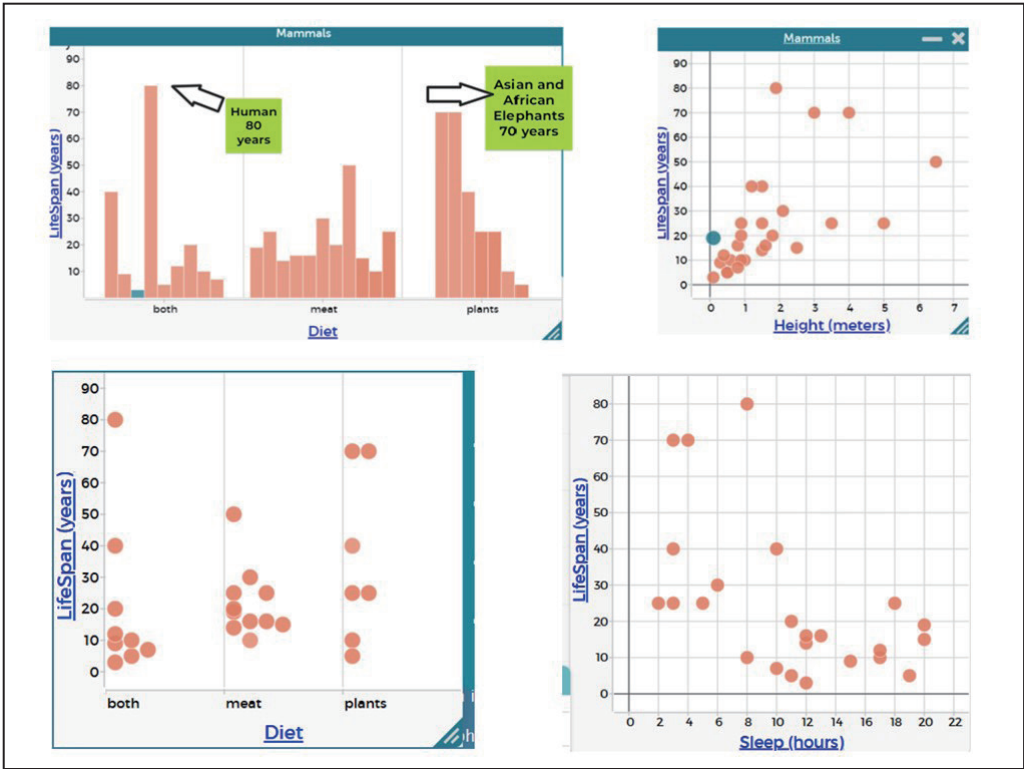


Figure 3. Data Visualizations of student work that are studying the effect of diet (top left and bottom left), height (top right) and sleep (bottom right)

Table 4. Instructional Shift in Data Literacy through technology-enhanced learning

Instructional Shift in Data Literacy	How to achieve it?
Data literacy instruction often involves passive consumption of information in a traditional way. However, with technology and visualization, there’s a shift towards active engagement. Encouraging students to a topic through technology-enhanced learning to actively interact with data through hands-on activities, exploration of interactive visualizations, and creation of their own visualizations using digital tools helps improve data visualization literacy.	<i>Moving to Technology-enhanced learning for Data Literacy</i>

Using technology and visualization effectively in creating tasks can help develop students’ abilities to interpret and create visual representations of data effectively (Mazzacane, Wiersma and Seshaiyer, 2023). This also allows teachers to shift assessment strategies from solely testing students’ knowledge of data concepts towards evaluating their ability to apply data analysis techniques and

communicate insights effectively. It also helps teachers to move beyond simply delivering information about data literacy concepts towards fostering inquiry-based learning experiences for students by posing open-ended questions and challenges that prompt students to investigate data, formulate hypotheses, and seek answers through data analysis and visualization. By embracing these instructional shifts, teachers can harness the potential of technology and visualization to transform data literacy instruction, empowering students to become proficient data analysts and critical thinkers in a digital and data-driven world.

Data Drives Durable Skills

When teachers employ instructional shifts and practices involving ideas like data talks in the classrooms or engaging students with no-code technologies such as CODAP to build student's data literacy identify it also helps to build an important lifelong data competency. This process not only helps to build foundational 21st century skills like critical thinking, communication, collaboration, and creativity for students (Suh and Seshaiyer, 2013), data literacy also helps to build character skills such as fortitude, growth mindset and leadership which helps to better integrate Durable Skills into education pathways that will help ensure a broader group of learners ultimately find success in their careers and communities.

Data literacy is one of those important durable and lifelong skills as well and may be described as the ability to understand and use data effectively to inform decisions. It is composed of a specific skill set and knowledge base that enables educators to transform data into information and ultimately into actionable knowledge (Mandinach et al., 2008). The research base supporting the characterization of data literacy skills and knowledge adequately demonstrates the complexity of using data, yet there is little agreement about data literacy as a construct or a competency (Mandinach & Gummer, 2012).

The focus on competencies in the past decades have been around 4 C's as life-long learning skills including Collaboration, Communication, Critical Thinking and Creativity. While these are used to design the goals, structure, organization and assessment of curricula; used by teachers and researchers to characterize good teaching and learning and; used in interactions with students, we posit these competencies we four more C's including Data in identifying Context, Data for building blended Curriculum, Data for integrating Content, Data for pedagogical Concepts which leads to Data Literacy for lifelong Competencies.

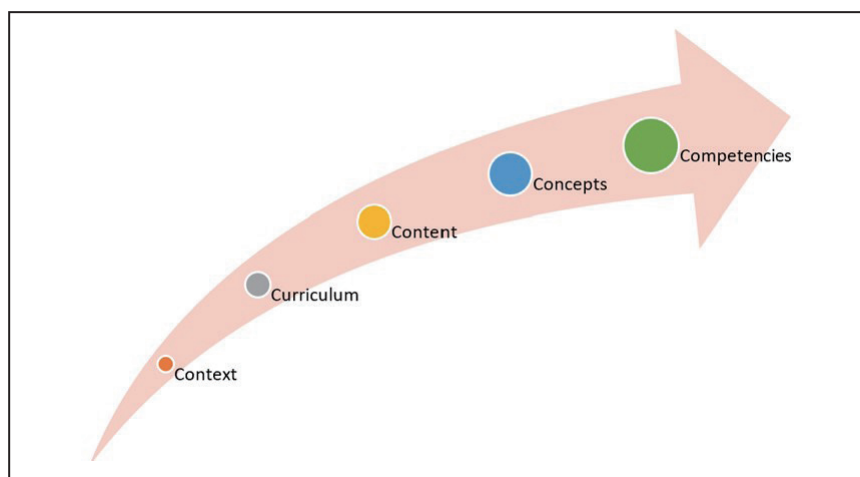


Figure 4. The (new) 5 C's

Table 5. Instructional Shift in Data Literacy through 5C’s

Instructional Shift in Data Literacy	How to achieve it?
Data literacy instruction needs to move from just focusing on finding the “right” answer towards emphasizing the process of data analysis and interpretation. Frameworks such as 5C’s support process-driven learning by encouraging students to focus on the steps they take to analyze data, rather than just the final results and in this way builds their durable skills and lifelong competencies.	<i>Moving towards a process focused approach</i>

Connecting data literacy concepts to real-world contexts enhances students’ understanding and relevance of the material. The 5C framework introduced here, can be used to contextualize data by having students notice real-world phenomena represented in data sets and prompting them to wonder about the implications of these observations.

Data Literacy as a framework convergence

Introducing students to data literacy at an early age allows them to incorporate inter- and intra-disciplinary approaches in mathematical thinking and the problem-solving processes. A good data literacy curriculum should include convergence of frameworks like statistical reasoning, mathematical foundations, computational thinking along with effective communication skills, and ethics. While these foundational frameworks tend to share certain similarities with data, they all aim to extract knowledge from data, for example—significant differences exist in the modeling processes, size of the data, type of problem, practitioners’ backgrounds, and language in question. Introducing concepts like statistical reasoning, mathematical foundations, computational thinking, and addressing the domain of the dataset from a young age can help develop a strong foundation in data literacy. Let’s break down each component:

- *Statistical Reasoning* helps us to understand basic statistical concepts such as mean, median, mode, range, and variance. This helps students to learn to interpret data through graphs, charts, and other visualizations. It even helps them to grasp the concept of probability and its role in data analysis.
- *Mathematical Foundations* help to develop a solid understanding of basic mathematical concepts like arithmetic, algebra, and geometry. Learning how to apply mathematical operations in the context of data analysis helps to understand mathematical models and their relevance in data interpretation.
- *Computational Thinking* helps to develop problem-solving skills through computational methods by helping students understand algorithmic thinking and how they can be used to process and analyze data. Learning to think systematically and logically to break down complex problems helps the students to be critical thinkers and creative problem solvers.

The convergence of these three frameworks and their ability to ask meaningful questions and draw insights from data prepares students to become data literate citizens who can apply their knowledge and skills effectively to a given context or domain such as health, business or social sciences. By integrating such data literacy components into the curriculum at young grades, students can develop a holistic understanding of data and its applications. This approach not only promotes data literacy but also enhances critical thinking, problem-solving, and analytical skills, which are valuable in various aspects of life and future.

A strong curriculum that integrates data literacy would ensure that students learn about how data is collected, processed, analyzed, interpreted, visualized, and shared with a variety of stakeholders. The experience should be project-based in that students collaborate with each other and their community to investigate questions of interest through data visualization, modeling, prediction, analysis, and communication. Student teams should be structurally similar to teams in the workforce, as every individual contributes their own assets to each project.

The transformation of mathematics education through data-driven pathways signifies a critical evolution in how we teach and learn. By leveraging data to guide instruction, integrate technology, and support the development of durable skills, educators can create more adaptive, inclusive, and effective learning environments. The convergence of instructional frameworks and data literacy reinforces the need for educators to not only interpret data but to act on it meaningfully. As this study highlights, the sustainable integration of data-driven practices holds the potential to elevate both student outcomes and teacher agency, laying the groundwork for a more responsive and future-ready mathematics education system.

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TEACHING MATHEMATICS THROUGH APPLICATIONS: A CASE OF MATHEMATICS EDUCATION FOR NON-STEM UNDERGRADUATE STUDENTS

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In recent years, as the areas in which mathematics is applied have expanded, undergraduate mathematics education for non-mathematics students has received more attention. Moreover, due to the growing importance of data science and statistics, mathematics education is now targeted not only at STEM students, but also at non-STEM students. However, mathematics is a difficult subject for many students to learn. In this context, there is a movement to reform the traditional style of mathematics education into a new style: the traditional style is to teach mathematics only in a mathematical context and then teach its applications later; the new style being explored is to teach mathematics using the context of students' major fields of study or of real-world situations. A famous example is the reform of a calculus course for life science students at UCLA. In this talk, I will explain the key points and difficulties of designing a course in this new style, using some examples, and I will also report on my past three years of practice in developing and implementing of a linear algebra course for non-STEM students using the context of statistics.

Keywords: undergraduate mathematics, non-STEM students, course design, linear algebra

Introduction

Teaching mathematics to non-mathematics students is receiving more attention in mathematics education at the university level (cf. Doukhan et al., 2024). In recent years, the range of students who need to learn calculus and linear algebra as a foundation has greatly expanded due to the expansion of the application areas of mathematics and the growing importance of data science, statistics, and AI. It has become necessary to consider mathematics education for students who have not traditionally been the target of mathematics education. In the era of data science and AI, the target of mathematics education at the university is no longer limited to STEM students. In the Japanese context, for example, the government is promoting “Mathematics, Data science and AI Smart Higher Education (MDASH)” for all students at the tertiary level (Cabinet Office, Government of Japan, 2021). In the model curriculum of MDASH, calculus and linear algebra are considered as fundamental subjects. Here, “all students at the tertiary level” means “all” university and college students, including non-STEM students to whom mathematics education is offered less or, in some cases, not at all. Teaching mathematics to non-STEM students is a challenging task because many non-STEM students are not good at mathematics and find mathematics a difficult subject.

The teaching of calculus and linear algebra in universities has always been problematic even for STEM students: calculus has become the gatekeeper for students to progress to STEM courses in many countries, with high dropout rates being a constant problem (Bressoud, 2016, Hochmuth et al., 2021); linear algebra remains a difficult subject to learn for many students due to its abstract nature (Dorier & Sierpinska, 2001). However, both the identification of these problems and suggestions for improving its teaching have traditionally been aimed primarily at mathematics, physics and engineering students (e.g., Kumar & Jalkio, 1999; Thompson & Harel, 2021). The teaching of linear algebra to non-STEM students has been little studied.

When teaching mathematics to students who struggle with the subject, a common approach is to minimize the content and simplify the explanations as much as possible. This is evident in the mathematics textbooks for these students. However, the serious problem for students who struggle with mathematics is not just that it is difficult. For them, the lack of understanding of the meaning and value of learning mathematics serves as a barrier to learning. For students who cannot find motivation to learn, even if the content is reduced and explanations are simplified, if the learning content remains abstract and its practical applications remain unclear, learning mathematics will remain a painful experience for them. Teaching mathematics in a way that helps students find meaning in learning it requires further effort. How can we help students find meaning in learning mathematics? To answer this question, it is important to consider not only what content to teach these students in mathematics (calculus and linear algebra), but also how to teach it.

In this article, we would like to discuss mathematics education especially for non-STEM students, that motivates them to learn mathematics and makes them understand mathematical concepts. The keyword of this article is “application”. In the following sections, we begin by discussing teaching style regarding the incorporation of application to mathematics teaching based on previous studies. Then, we observe some examples of teaching mathematics through applications. In the latter half of the article, we report on the development and implementation of a linear algebra course for non-STEM students in which the author was involved. Finally, we discuss the difficulties and future issues in implementing mathematics education through applications.

Raison D’être of Knowledge, Monumentalism, and Applicationism

As a key to motivating students who struggle with mathematics, we would like to highlight the notion of “raison d’être” of knowledge, which Chevallard often refers to (e.g., Chevallard, 2006). According to Chevallard, any knowledge is created for a purpose. The fact that it was created for that purpose is precisely its *raison d’être*. Chevallard (2006) argues that conveying the *raison d’être* of knowledge is crucial in mathematics education and criticizes education without conveying the *raison d’être* of knowledge as “monumentalistic.”

There is an easy way to make the current, “monumentalistic” school epistemology visible – by asking for the reasons why such and such praxeology or such and such praxeological “ingredient” exists. Why do mathematicians seem so attracted to triangles example? Why does geometry tell us about angles, lines and rays, or about crossing lines and parallel lines? Why does geometry make room for the notions of acute angle, obtuse angle, and reflex angle? If you are tempted to answer: “Mathematicians are interested in all these entities simply because there do exist crossing lines, rays, acute angles, reflex angles, etc., that is, just because these ‘things’ are out there, in the natural world, waiting for us to study them”, then you have been infected with the evil “monumentalistic” doctrine that pervades contemporary school epistemology. (Chevallard, 2006)

Asking what mathematical knowledge is used for is the same as asking what it can be applied to. Of course, this includes applications within mathematics itself, such as applying it to more advanced mathematical problems or to other branches of mathematics. However, when considering mathematics education for non-mathematics students, applications outside mathematics are more important. For example, regarding a mathematics curriculum for engineering students, Alpers et al. (2013, p. 58) stated that “the mathematics curriculum should be strongly linked with the application subjects taught in the engineering study course under consideration.” Indeed, there are mathematics textbooks that introduce many applications to engineering and real-world problems with the aim of motivating students. However, there is an issue to consider when incorporating applications into

mathematics education. This is “applicationism” (Barquero et al., 2011; 2013).

Barquero et al. (2011) call the dominant epistemology in university education, which sees mathematics and the other disciplines to which it is applied as not influencing each other, “applicationism” and characterize it by the following indicators.

I₁: Mathematics is independent of other disciplines (‘epistemological purification’): mathematical tools are seen as independent of extra-mathematical systems and they are applied in the same way, independently of the nature of the system considered.

I₂: Basic mathematical tools are common to all scientists: all students can follow the same introductory course in mathematics, without considering any kind of specificity depending on their speciality (biology, geology, etc.).

I₃: The organisation of mathematics contents follows the logic of the models instead of being built up from considering modelling problems that arise in the different disciplines. All occurs as if there were a unique way of organising mathematical contents and different ways of applying them.

I₄: Applications always come after the basic mathematical training: the result is then a proliferation of isolated questions that have their origin in the different systems. The first thing is to learn how to manipulate the mathematical concepts and later learn about their use. The models are built from concepts, properties and theorems of each theme independently of any extra-mathematical system.

I₅: Extra-mathematical systems could be taught without any reference to mathematical models, that is, there exists the belief that natural science can be taught without any mathematics. (Barquero et al., 2011)

Barquero et al. (2011) used the concept of “applicationism” to identify the constraints that make mathematical modeling education difficult, but their conclusions are highly suggestive for mathematics education at the university level.

One of the main characteristics of ‘applicationism’, which represents one of the strongest restrictions on the normal life of mathematical modelling, is that it establishes a clear distinction between mathematics and the rest of natural sciences. It is furthermore supposed that both „worlds” evolve with independent logic and without many interactions. [...] This radical separation between mathematics and natural sciences makes it very difficult for students to consider mathematics as a constituting tool of natural sciences and, thus, to value the necessity of its learning. (Barquero et al., 2011)

Applicationism is observed both among mathematics teachers (i.e. mathematicians) who teach mathematics in universities and among teachers in specialized fields (such as engineers, who are not mathematicians) where mathematics is used. As a result, a disconnect has emerged between the mathematics taught in general education and the mathematics used in specialized fields. This has been demonstrated in several studies (e.g., González-Martín, 2021; Hitier & González-Martín, 2022).

Three Styles of Mathematical Teaching from the Viewpoint of Incorporating Applications

Based on the discussion in Barquero et al. (2011), the style of mathematics teaching can be broadly classified into the following three categories according to differences in the stance taken toward the incorporation of applications.

Teaching mathematics only in mathematical contexts (MM): In this style, regardless of whether mathematics is taught theoretically through a series of definitions, theorems, and proofs, or only through the teaching of computational procedures, applications are limited to problems within the intra-mathematical context, and applications in extra-mathematical contexts are not mentioned.

Teaching mathematics first and applications later (MA): In this style, mathematics is taught basically in the MM style, but then it deals with application problems in extra-mathematical contexts.

Starting from application situations to teach mathematics (AM): In this style, in contrast to MA, teaching begins with practical applications, and various mathematical concepts appear as tools necessary for these applications.

In traditional mathematics courses such as calculus and linear algebra, the content is almost identical to that of courses for mathematics students, and applications in extra-mathematical contexts are often not taught (MM style) or postponed until later (MA style). The MM style is dominant in Japan. Textbooks for mathematics courses (such as calculus, linear algebra, differential equations, etc.) used in most Japanese universities are written in this style. Most mathematics courses in Japan are characterized by the indicators I_1 , I_2 , and I_3 of Barquero et al. (2011).

The MA style can be found in standard textbooks used in the U.S. (e.g., Anton & Kaul, 2019; Larson, 2016; Hughes-Hallet et al., 2013; Stewart et al., 2020). These textbooks include examples and exercises in extra-mathematical contexts, but mostly at the end of each chapter. This style is characterized by the indicator I_4 of Barquero et al. (2011). The MM and MA styles of teaching have long been pointed out to be problematic in terms of students' motivation to learn mathematics, as they lead to a separation between mathematics and its applications.

We have classified the mathematics teaching styles into the three styles, but this does not mean that every mathematics course matches only one style. A course that is basically MA style may be partially AM and vice versa. However, there are very few practices whose design principles are based on the AM style. Barquero et al. (2011) considered this style in the context of mathematical modeling education. The report of Viirman and Nardi (2019) on mathematical modeling sessions conducted in parallel with regular mathematics courses for biology students can also be included as an example of AM-style practice. Among the mathematics courses, we can find some examples, which we show in the next section.

Examples of Mathematics Education in the AM Style

Example in general education mathematics courses for non-STEM students

The author has been involved in the development and implementation of a mathematics course for first-year non-STEM students since 2010 (Kawazoe et al., 2013; Kawazoe & Okamoto, 2017; Kawazoe, 2022). It should be noted that this was a collaborative work between a mathematician (the author) and a cognitive psychologist. The fact that this course is AM style is clearly illustrated by the following statement in the preface of the textbook (Kawazoe & Okamoto, 2012) for this course:

In general, mathematics textbooks up to high school level introduce new mathematical concepts, how to solve questions or formulas at the beginning, and then the student learns these concepts and solutions by solving example problems. However, since the aim of this book is to develop the ability to think about real-world situations using mathematics, each chapter begins by presenting real-world problems to be considered and then explaining how to think about such problems using mathematics. [...] In this book, after learning to think about real-world problems using mathematics, mathematical

definitions and formulas are explained as a summary. (Kawazoe & Okamoto, 2012, pp. ii–iii. Translated by the author from Japanese.)

This course is designed as a 2-semester mathematics course for non-STEM students precisely to address their difficulties with mathematics and uses the context of mathematical modeling to achieve AM-style teaching. The course started as *Basic Math I & II* from the 2012 academic year, and it has been renamed as *Mathematical Literacy 1 & 2* from the 2022 academic year. At the beginning of each lesson, a real-world problem is presented in its unmathematized form, and the mathematical knowledge used for modeling the problem emerges through the mathematical modeling activity. The mathematical content and the application context of the course are shown in Table 1.

Table 1. The mathematical content and the application contexts of *Mathematical Literacy 1 & 2*

Mathematical content	Application contexts
systems of linear equations/inequalities	linear programming
number sequences	savings, loan payment, pharmacokinetics, model of addiction
matrices and vectors	spreadsheets, social network analysis
functions	mental rotation, pharmacokinetics, bacterial growth, pandemic, periodic movement of electric demand
probability	lottery, disease examination, birthday paradox, Bayesian estimation
eigenvalues/vectors	population dynamics, optimal distribution
derivatives	innovation diffusion, population growth, logistic function, marginal profit, optimization
integrals	speed and distance, accumulated radiation level, standard normal distribution
multivariable functions	loan simulator, formulas for estimating vital capacity, optimization

In Table 1, the first five content items are for the spring semester (Mathematical Literacy 1) and the remaining four are for the fall semester (Mathematical Literacy 2). Much of the mathematical knowledge that emerges in mathematical modeling in the course comes from Japanese high school mathematics. However, matrices including eigenvalues and eigenvectors, partial derivatives and min-max problems for two-variable functions including Lagrange multiplier method, which are not currently covered in the Japanese high school curriculum, are also designed to be taught through mathematical modeling activities using real-world problems.

The design principles for the course and various examples of real-world problems used in the course are presented in (Kawazoe & Okamoto, 2012; Kawazoe, 2022). Here, we present only the problem used to teach eigenvalues and eigenvectors (Figure 1).

Problem. *O University* has decided to start a rental bicycle service because of its huge campus. Two parking areas will be located beside the main gate *Nakamozu-mon* (Parking Area A) and in front of the lecture building *Building B3* (Parking Area B). Students can drop bicycles off at any of the two parking areas. A monitoring investigation showed that bicycles at both parking areas move as in the following table every week. Numbers in the table are displayed as percentages. How should we divide bicycles between the two parking areas when the service starts?

	To Parking Area A	To Parking Area B
From Parking Area A	70	30
From Parking Area B	20	80

Figure 1. Rental bicycle service problem (Kawazoe, 2022)

The lesson using this problem will take place over two 90-minute sessions. Students have already learned matrix operations and the concept of transition matrices, but they will enter this lesson without having studied eigenvalues and eigenvectors. First, students will look at the numbers in the table shown in the problem and discuss in groups whether it is better to allocate more bicycles to A, more to B, or distribute them equally to both. Then, each group will be given a different number of bicycles and asked to consider a specific initial allocation based on their predictions. Then they calculate the changes in the number of bicycles over the next five weeks using the numbers shown in the table. Students can use calculators and computers to make these calculations. The lecturer collects the results from all groups and displays them on a graph (Figure 2).

Here, it can be observed from Figure 2 that the points representing the number of bicycles in the two bicycle parking areas move along parallel straight lines and converge to a single straight line passing through the origin (the blue line in Figure 2). By examining that the vectors representing the directions of the two lines are related to the transition matrix, the concepts of eigenvalues and eigenvectors are introduced. Furthermore, through the exercises, the students learned that the transition of the number of bicycles can be explained using eigenvalues and eigenvectors. In this lesson, students encounter eigenvalues and eigenvectors in a real-world situation and learn both the meaning and the usefulness of these concepts.

Many students taking this course have a negative attitude toward mathematics. However, a survey conducted in the 2012 academic year, when the course began, showed that the course increased students' interest in mathematics and improved their confidence in the subject (Kawazoe et al., 2013). The survey was conducted at the end of the semester and asked the students "How has your interest in mathematics changed after taking this course?" and "How do you think your ability to think using mathematics has changed after taking this course?" 60% of the students responded to the first question that their interest had increased ("significantly increased" and "increased"), while very few students responded that their interest had decreased or significantly decreased. Over 70% of the students responded to the second question that it had improved ("significantly improved" and "improved"), while very few students responded that it had decreased or significantly decreased.

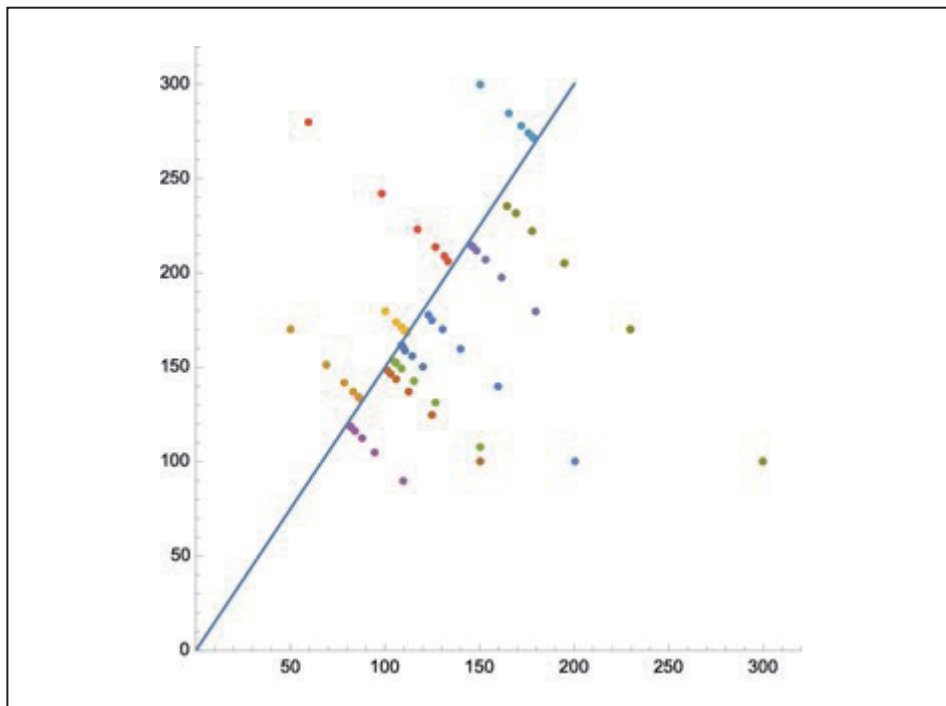


Figure 2. The plotted graph showing the transitions of the number of bicycles

Example in mathematics courses for economics and management students

Another example in Japan is found in mathematics courses for economics and management students. The textbook written by Oyama and Yamada (2013) is a good example of an AM-style textbook for such a course. This book is written by a team of economists. Each chapter of this book starts with a real-world problem in the context of economics or management. As for mathematical topics, this book covers the following: function (linear, quadratic, exponential), logarithm, number sequence, derivative of single and multiple variables functions, vectors and matrices, probability, and integral of single variable functions. This book is used in several Japanese universities as a textbook or as a reference book for mathematics courses for economics and management students. The preface of the textbook (Oyama & Yamada, 2013) shows that this book is written in the AM style:

Therefore, in this book, instead of introducing mathematical concepts right away, we have taken the approach of starting with economic topics and then learning the mathematics used in those topics. For example, in Chapter 1, which deals with linear functions, we use a publisher's market research as an example, and in Chapter 2, which deals with quadratic functions, we use a ramen shop's profit maximization as an example. In this way, we first present concrete examples and then explain the mathematical concepts used in these examples, ensuring that the importance of using mathematics to solve economic problems is easily understood throughout the text. (Oyama & Yamada, 2013, p. i. Translated by the author from Japanese.)

The approach of this book is very similar to that of Kawazoe and Okamoto (2012), but differs in that Kawazoe and Okamoto (2012) present real-world problems in their unmathematized form, while Oyama and Yamada (2013) explicitly provide mathematical models to be used in many problems (Figure 3). This difference can be seen as a difference in setting up the modeling process as a

student activity from what point on. Despite this difference, they have in common that they present problems and solutions using a mathematical modeling framework.

Example. You are the owner of a ramen restaurant chain, and you are considering opening a new ramen restaurant in a shopping mall in City A. There are no other ramen restaurants in this shopping mall, and the demand for ramen is given by the demand function $D(p) = 500 - 1/2 p$ (where $0 \leq p \leq 1000$). In addition, the cost of producing one bowl of ramen, including labor costs, is exactly 400 yen.

- (1) Calculate the profit of the ramen shop in two different ways: as a function $f(p)$ of price p , and as a function $g(q)$ of production quantity q .
- (2) What should be the price per bowl in order to maximize profit?
- (3) In addition to the above costs (400 yen per bowl), you have to pay 1.2 million yen in rent each month. Should you open the ramen shop? (Assume that a month has 30 days.)

Figure 3. A ramen restaurant problem (Oyama & Yamada, 2013, p.33)

UCLA curricular reform of calculus for life science students

Among examples of the AM style, the reform of the calculus course for life science students at UCLA is noteworthy in that it is a reform of a standard regular mathematics course. Details of the course reform and its outcomes can be found in articles reporting on the reform (Burdman et al., 2021; Burdman, 2022; Mollenkamp, 2023) and in reports by Garfinkel et al. (2022), who were directly involved in the reform. In addition, the published textbook (Garfinkel et al., 2017) and videos available online (UCLA modeling class, 2019) provide detailed insights into the course content. It should be noted that this reform was accomplished by faculty members in the life sciences. Here, we provide a brief introduction to this UCLA reform case.

The development of a new calculus course at UCLA was prompted by issues with the previously offered “Calculus for Life Sciences” course, which lacked connections to the life sciences and undermined the motivation of life science students, thereby reducing the number of students aiming to pursue STEM fields. The guidelines for developing the new course were based on reports from National Research Council (2003), American Association of Medical Colleges and Howard Hughes Medical Institute (2009), and Brewer and Smith (2011). The preface of the course textbook (Garfinkel et al., 2017) states that the course aims to incorporate all these suggestions.

Our approach was to incorporate ALL of the above suggestions into our course and into this text. We “study nonlinear dynamical systems, featuring positive or negative feedback.” We “explain how feedback mechanisms lead to phenomena like switch-like oscillations.” We study examples like “dynamic modeling of neural networks and dynamics in ecology.” Overall, our approach is to “use ordinary differential equations, made tractable and understandable via Euler’s method without any formal course in differential equations.” (The quoted phrases are directly from the above publications.) (Garfinkel et al., 2017, p. vi)

This newly developed course for life science students focuses on biological problems, beginning with the presentation of a biological problem and then modeling it. The course emphasizes the necessity of modeling in pursuing biological problems, and mathematical concepts are introduced

based on the reasons why they are necessary for modeling. The mathematical content is structured in a sequence driven by modeling biological problems. In other words, the course is designed so that mathematics emerges from the need for modeling, rather than being introduced independently before application.

The modeling of biological problems is always at the center of the course. Moreover, rather than presenting different independent topics in each chapter or section, a single context is provided across chapters to offer a larger learning path. The discussion of population dynamics in a predator-prey ecosystem introduced at the beginning of the first chapter (a graph showing the oscillations in the populations of lynx and snowshoe hares over time is presented at the beginning) leads to the study of functions, state spaces, vector fields, derivation, integration, and differential equations. The exploration of the predator-prey system serves as the core, providing the overarching learning path, with related topics arranged in a logical sequence along this path.

We should emphasize that these examples are not simply a form of window dressing meant to make a particular set of mathematical approaches palatable to students. Rather, the class is structured around the idea that, as biologists, we are naturally interested in understanding these kinds of systems. In order to do that, we need to develop a mathematical framework for making, simulating and analyzing dynamical models. Using these biological systems not purely as examples, but rather as the core motivation for studying mathematical concepts, provides an intellectual framework that deeply interests and engages life science students. (Garfinkel et al., 2022, p. 18)

In the latter half of the text, the predator-prey system also provides a context for learning about eigenvalues and eigenvectors in linear algebra. Garfinkel et al. (2022) state that they took the shortest learning path to eigenvalues and eigenvectors (Garfinkel et al., 2022, p. 11), which is nearly identical to the approach taken by Kawazoe and Okamoto (2012).

Garfinkel et al. (2022) refer to the results of a study (O’Leary et al., 2021) that evaluated the effectiveness of this course. According to the survey, 79.7% of respondents reported gaining confidence in mathematics, 74.8% stated that the course helped improve their science skills, and 90.8% expressed a desire to take more life science courses. This contrasts with the findings that many students who had taken at least part of a traditional “Calculus for Life Sciences” course reported a decline in confidence in their mathematical abilities and did not agree that they had become more interested in biology after taking the course. Among the above results, Kawazoe et al. (2013) also reported similar findings regarding confidence in mathematics in their AM-style teaching practice, indicating that the AM-style approach is promising.

Characteristics of the AM Style Courses

Let us summarize the common features of the three examples of AM-style teaching presented above. What the three AM style examples have in common is that they all start with an application problem with an extra-mathematical context, mathematics is introduced through the mathematical modeling necessary to approach this problem, and the explanation of mathematical definitions and properties comes at the end. This is emphasized by the authors of the respective textbooks. The AM style of teaching does not simply motivate students by showing application problems at the beginning and by declaring that the mathematics to be learned is useful in these problems. The essence of the AM style is to “teach mathematics through the context of applications”. The way of choosing application problems is also common. The application problems used in each course or textbook are chosen to be of interest to the learners, who are assumed to be the targets, or to be related to their future learning. It is particularly evident in the third UCLA case that a single or a set of related

application problems become the core of a learning path, either across an entire chapter or beyond a chapter.

From the above observation, we can present a guideline for designing an AM-style course as follows:

1. Start with applications and teach mathematics through the context of applications.
2. Include a mathematical modeling perspective.
3. Develop learning paths for the course with core application problems.
4. Ensure that core application problems stimulate learners' motivation to learn and lead to further learning (e.g., in a field of students major).

Developing a Linear Algebra Course for non-STEM Students with a Link to Statistics

In this section, we report on the development of a new linear algebra course for non-STEM students in which the author was involved as a designer and lecturer (cf. Kawazoe, 2024). This new linear algebra course was created in response to a request from the Psychology Department for a new 2-semester linear algebra course with a link to statistics for their students. In response to this request, we thought it would be appropriate to design this new linear algebra course in the AM style. This was because, as the first and third examples in the previous section showed, AM style teaching seemed to contribute to motivating students who were not good at mathematics and less motivated to learn mathematics, and to increase their confidence in mathematics. However, we were unable to find any linear algebra textbooks written using this approach. Since the late 1990s, many linear algebra textbooks have been published that aim for applications in statistics and data science. The book by Harville (1997) covers selected topics relevant to statistics, but does not discuss applications. The book by Banerjee and Roy (2014) is the MM style with a very small number of examples related to statistics. The book by Gentle (2007) refers to statistics but follows the MA style approach. The book by Strang (2019) emphasizes applications to data science, but is too advanced for non-STEM students. The book by Garfinkel et al. (2017) follows the AM style approach in the calculus part, but the linear algebra chapter seems to be written in the MA style. Many Japanese textbooks also follow a similar pattern (e.g., Shiina et al., 2019). The textbook for non-STEM students by Kawazoe and Okamoto (2012), which is used for the course of the first example in the fourth section, is written in the AM style and introduces matrices, eigenvalues, and eigenvectors from application situations, but it only deals with a part of linear algebra and does not cover the entire linear algebra course. Therefore, we decided to design this new linear algebra course from scratch, referring to the AM-style examples introduced in the fourth section.

Strategy for developing the new course

The purpose of this project was to develop a linear algebra course that is particularly suitable for first-year non-STEM students at university, with the learning context of applications to data science and statistics: more precisely, to do for linear algebra in the context of data science and statistics what Garfinkel et al. (2017) did for calculus in the context of biology; to extend our previous attempt (Kawazoe & Okamoto, 2012) to the entire linear algebra course. To develop the course, we had to address the following questions. (Q1) How can we make sense of the concepts and computational methods of linear algebra from the perspective of statistics and data science? (Q2) What concrete learning contexts are possible (and appropriate) for non-STEM students to learn linear algebra? We addressed these questions in the following three steps: (1) analyze how linear algebra

concepts relate to tasks or methods in data science and statistics; (2) develop concrete learning contexts based on the results of the previous step; (3) examine students' reactions in the course. In the following, students in the course have already studied planar and spatial vectors in high school but have not studied matrices.

First step: Analysis of how linear algebra concepts relate to data science and statistics

Before describing the results of the first step, we mention the constraints and limitations of this course. Since this course was to be developed as a reform of the existing linear algebra course for STEM students, there was a constraint that the learning units of this course should basically follow the course for STEM students. Due to this constraint, the learning units of each semester (spring and fall) consist of the following: vectors and matrices, operations on vectors and matrices, systems of linear equations and Gaussian elimination, computing inverse matrices with Gaussian elimination, determinant (spring semester); basis and dimension of vector space, linear map and its kernel and image, diagonalization, inner product space, orthogonal diagonalization (fall semester). The limitation of the course is the notion of vector spaces to be taught. To design a linear algebra course linked to the learning of statistics, we decided to adopt the principle of looking at vectors and matrices in such a way that they are forms of representation of data and their operations can be viewed as data processing. Following this principle, vector spaces in the course are limited to numerical vector spaces.

To carry out the first step, we examined various books and information on the Internet related to statistics and data science, and matched the mathematical concepts and calculations that appear in statistics and data science with the content of linear algebra. Then, we carefully selected topics so that first-year non-STEM students could understand them, and so that the list of selected topics could provide learning paths. Since the topics of data science are too advanced, we focus on statistics and data processing, which is much more elementary than data science. The result of this step is shown in Table 2. Table 2 shows the correspondence between the content of linear algebra and the content of statistics and data processing.

Table 2. Relation between linear algebra concepts and the content of statistics and data processing

Concepts in linear algebra	Meaning from statistics and data processing
Vector (sum, scalar multiplication, linear combination)	Pairs of data (related calculations on data)
Inner product	Weighted sum, covariance
Vector space	A data space
Matrix (sum, scalar multiplication)	Table of data (related calculations on data)
Multiplication of matrices	Aggregating from two tables
Diagonal matrix	Constant multiplication per table column/row
Power of matrix	Markov chains, population dynamics, etc.
Transposition of matrix	Transposition of table
Systems of linear equations	Finding a set of data satisfying a given condition

Inverse matrix	Reversing a Markov chain process
Determinant	Finding eigenvalues for Markov chain, etc.
Basis, dimension	Degree of freedom in space
Linear map	Map to another data space
Image of linear map	A range of the results (outputs)
Kernel of linear map	Changing inputs with the same result (output)
Orthogonal projection	Linear regression model
Eigenvalues/vectors, diagonalization	Markov chains, population dynamics, PCA, etc.
Orthogonal diagonalization	Principal component analysis (PCA)

To develop learning paths for the course, rather than presenting independent learning paths in small chunks, we considered covering the entire course with as few learning paths as possible by ensuring that each learning path covers as many learning units as possible. The learning paths that we have developed from Table 2 are as follows:

1. Basic computations for the data processing introduce operations on vectors and matrices (sum, scalar multiplication, linear combination, inner product of vectors, multiplication of matrices) and some special matrices related to some computations.
2. The multiplication of a matrix by a vector leads to the study of $Ax=y$ and then derives three branches from the previous path:
 - a. Finding a x data that satisfies the condition $Ax=y$ leads to the study of systems of linear equations and further leads to the study of basis and dimension by considering solution spaces,
 - b. Viewing the equation $Ax=y$ as a function with an input data x and an output data y leads to the study of linear maps and further leads to the study of the image and kernel of a linear map,
 - c. Markov chains and population dynamics lead to the study of the power of matrices, inverse matrices, eigenvalues, eigenvectors, and diagonalization.
3. The inner product of vectors derives another branch of linear regression and principal component analysis (PCA), and leads to the study of distance, orthogonal projection, orthonormal basis, and orthogonal diagonalization.

Second step: Developing concrete learning contexts

In the second step, we developed concrete learning contexts suitable for the learning paths proposed in the previous step. First, we need to make sense of generalizing the notion of vectors to higher dimensions, because non-STEM students entering our university have learned planar and spatial vectors in high school but have not learned higher dimensional vectors and matrices. Our principle of viewing vectors and matrices as a tool for representing data gives rise to the need to generalize planar and spatial vectors to higher dimensional vectors, because vectors and matrices are

understood as numerical data arranged in rows/columns. However, we need a concrete context for using higher dimensional data that can be used in a large part of the learning paths. In developing this context, we were inspired by the “Making Granola Task” (Figure 4), which has been used in mathematical modeling education (Cristóbal-Escalante & Vargas-Alejo, 2013). The context of making granola (Figure 4) can cover learning paths 1, 2a, and 2b presented above, but we changed “granola” to “vegetable juice”. This is because granola is less familiar in Japan, and vegetable juice provides a natural context for working with vectors of continuous values.

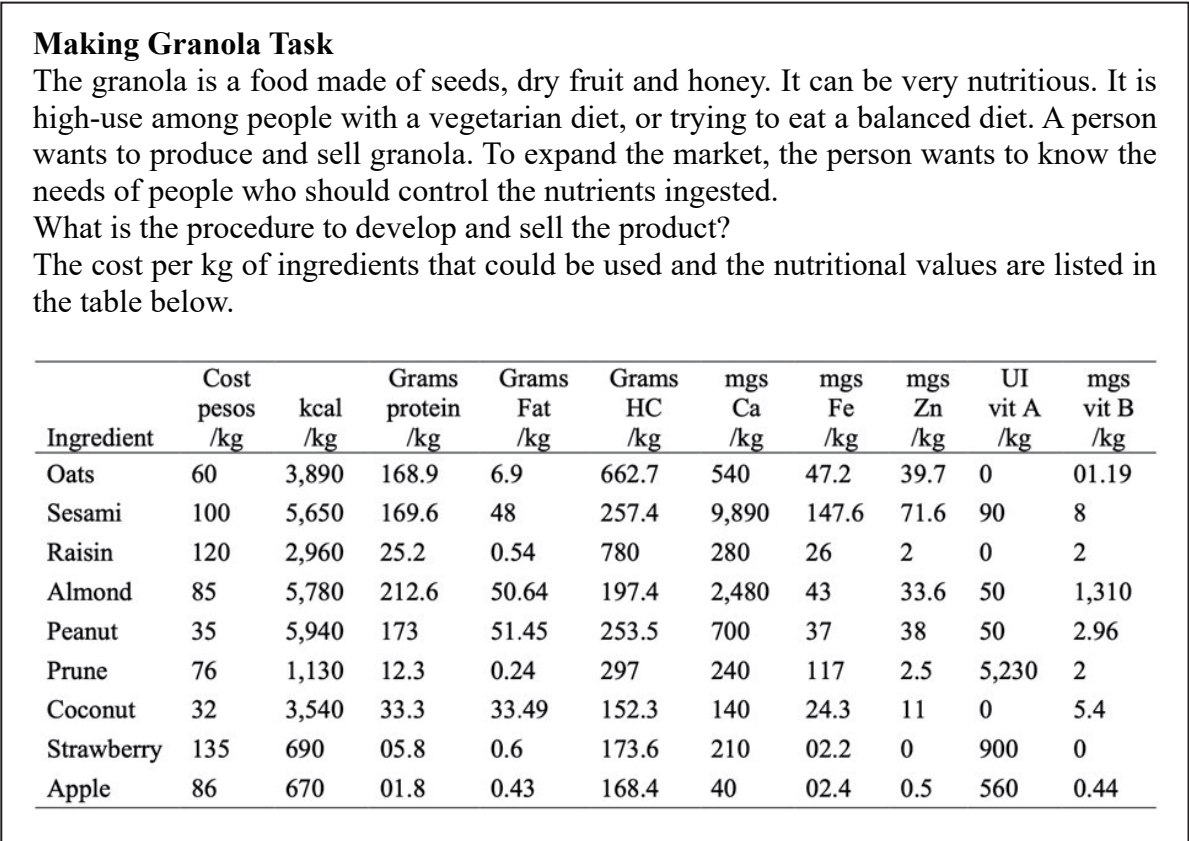


Figure 4. Making granola task (Cristóbal-Escalante & Vargas-Alejo, 2013)

Vectors of four or more dimensions appear naturally in the situation representing the ingredients (e.g., energy, vitamin C, carotene, calcium, etc.) per 100g of each vegetable for making a vegetable juice. The sum, scalar multiplication, and linear combination of vectors appear in the calculation of the ingredients of a juice made from various amounts of vegetables. The inner product appears as a calculation to find the total amount of an ingredient from the data of the amounts of vegetables and the data of the ingredient per 100g of each vegetable.

Matrices can be introduced from tables of vegetables and ingredients, but for the introduction of matrix operations, we chose another situation that represents a table of the number of units sold of several products in several stores on a given day. This is because the context of sales is familiar and easy for students to understand. The sum of matrices appears in the calculation of the total number of each product sold over several days. Scalar multiples appear in the calculation of the total sales when, for example, sales are doubled. The product of matrices appears when we want to calculate

a table of total sales and total profit per store from a table of units sold and a table of the price per product and the profit amount per unit.

Moreover, the product of matrices used for a special situation provides a meaning for learning special matrices. For example, to find a table of the sales amount of each product in each store, multiplication by a diagonal matrix is used. In this situation, diagonal matrices appear as tools for finding the scalar multiples of each row by different numbers and for finding the scalar multiples of each column by different numbers.

The product of a matrix and a vector can be given in several contexts. One context is calculating the amount of each ingredient in a vegetable juice from a table of the amount of ingredients per 100g of vegetables and a vector of the amounts of vegetables used. In this context, the following questions naturally arise:

1. Compute the amount of each ingredient obtained by making vegetable juice from a given set of vegetables.
2. Find the amount of each vegetable to make a vegetable juice that meets a given list of ingredient quantities.
3. Is it possible to change the amount of vegetables without changing the amount of ingredients?
4. What combinations of the amount of ingredients can be obtained?

Question 1 provides a context for the product of a matrix and a vector. Questions 2-4 can be used as a context for learning systems of linear equations. Viewing the amount of vegetables as input and the amount of ingredients contained in a vegetable juice obtained as output, the context of making vegetable juice can be regarded as a linear map from a space of vegetables to a space of ingredients. Then, the above four questions can be used as a context for learning linear maps and their kernels and images. In this context, we can make sense of the basis and dimensions as a tool to capture the size of the space of images and kernels of linear maps.

Another context is a Markov chain. The Markov chain can be used as a context for learning the powers and inverses of matrices, eigenvalues, eigenvectors, and diagonalization. Several concrete problems are used in this course. Changes in market share over a period of several years are used to introduce matrix powers. The rental bicycle service problem (Figure 1) presented in the fourth section is also used in this course to introduce the concepts of eigenvalues, eigenvectors, and diagonalization.

Inner product space, orthogonalization, and orthogonal diagonalization can be learned using linear regression with the least squares method and principal component analysis (PCA) as the learning context. With the introduction of the fact that error minimization is related to orthogonality, by extending the relationship between inner product and orthogonality to four or more dimensions, orthogonal projection, orthonormal basis, and the Gram-Schmidt orthogonalization method can be introduced with necessity. The inner product also appears in the covariance, and orthogonality is related to the zero covariance and the zero correlation. Furthermore, since PCA is mathematically nothing but the orthogonal diagonalization of a variance-covariance matrix, it can be used as a context for learning the orthogonal diagonalization of a symmetric matrix. As a concrete example of PCA, we chose the problem of creating a comprehensive evaluation score from a list of evaluation scores for noodles and soup, and comparing several “udon” restaurants, which is used in the first example in the fourth section (Kawazoe & Okamoto, 2012). However, due to the time constraints, the teaching of orthogonal diagonalization is rather MA style at this moment.

Third step: Examining students' reactions in the course

We implemented the above contexts in a new 2-semester linear algebra course beginning in the 2022 academic year. In each semester, we collect students' responses to each week's lesson, which consist of free-form written comments and the level of difficulty by using a 5-point Likert scale (1=very easy, 5=very difficult). Figure 5 shows the average score of the students' responses to the 5-point Likert scale item each week in the 2024 academic year (week 1-15 in the spring semester and week 1-14 in the fall semester). In the 2024 academic year, 112 students took the spring semester course and 73.8% of them responded to the item each week on average. Furthermore, 81 students took the fall semester course and 70.7% of them responded to the item each week on average. Since there is no week in which the average score exceeds 4, it can be said that the difficulty of the course is appropriate for non-STEM students. In particular, the fact that the average scores were kept below 3.5 during weeks 6-10, where students learned eigenvalue, eigenvector, and diagonalization through the rental bicycle service problem (Figure 1), indicates that the context of these weeks was successful.

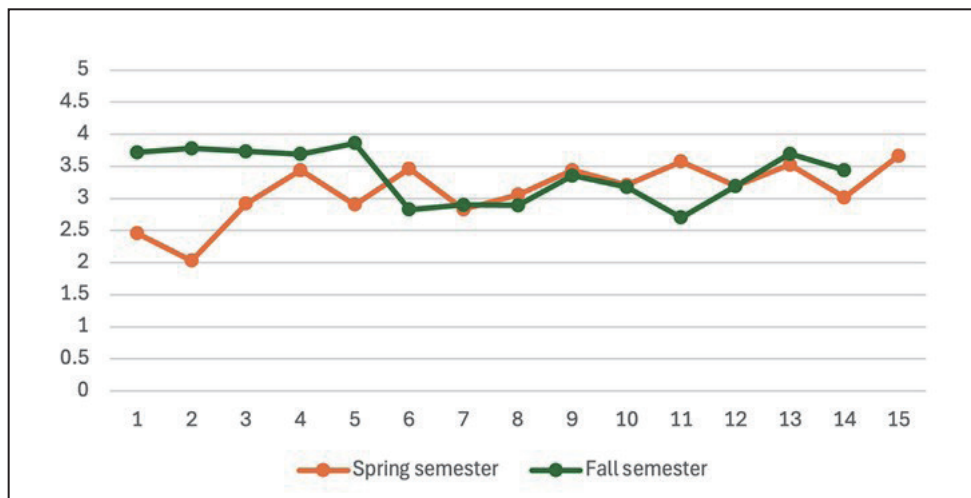


Figure 5. The students' responses to the difficulty level of each week's lesson

In the survey conducted at the end of the 2024 academic year, we included the following three 5-point Likert scale items to examine how students felt about the use of contexts and applications of statistics and applications in both semesters of the course:

1. Did the explanation of statistics and applications help you understand the importance of learning linear algebra?
2. Were the statistics and applications of linear algebra presented in the class interesting?
3. What was the difficulty level of the explanation of the statistics and applications of linear algebra presented in the class?

56 of the 81 students responded to these items. Many of the students responded positively to the first two items: For item 1, 82.1% answered that it was (very) useful and only 3.6% answered that it was useless; for item 2, 67.9% answered that it was (very) interesting and 10.7% answered that it was not interesting. On the other hand, for item 3, 75.0% of them answered that it was (very) difficult. The result of item 3 may seem to contradict the result of the weekly difficulty level. Further

investigation is needed, but the fact that some statistical tools (e.g., correlation coefficient, linear regression model, PCA) were not sufficiently implemented in the course and thus functioned as extra topics rather than learning contexts may have contributed to the students' perception of difficulty. For the context of using vegetable juice and the rental bicycle service problem, students' positive responses were found in their free comments collected weekly: e.g., "Through the specific example of vegetable juice, I learned the way of viewing space and the situation where we can use linear maps"; "Because the lesson was in a style of thinking mathematically about real-world problems by using a bicycle example, I found it very interesting. The explanation of 'eigenvalues' using a concrete example was very easy to understand."

Discussion

In this section, we discuss the issues of developing an AM-style course based on our attempt to develop a linear algebra course for non-STEM students presented in the previous section.

Context development issues

The responses of the students in the survey conducted in the 2024 academic year that we showed in the previous section indicate that the contexts used in introducing the concepts and operations on vectors and matrices in teaching systems of linear equations, linear maps, eigenvalues, eigenvectors, and diagonalization are effective. In this sense, most of the contexts seem suitable for non-STEM students to learn linear algebra. However, we have not been able to provide an effective context for learning inner product space, orthogonal projection, and orthogonal diagonalization. The goal of learning inner product space is to learn orthogonal diagonalization. Principal component analysis (PCA) is mathematically an orthogonal diagonalization, but it may be too difficult for the students. The current situation that many students find the contexts in the orthogonalization and orthogonal diagonalization part difficult should be improved. At the moment, we have not been able to develop a learning path with an effective context using PCA. We need to consider when and how to present a concrete problem where PCA is applied in the teaching of inner product spaces.

Curriculum constraint

As mentioned in the previous section, the development of the new linear algebra course was done under the constraint that the learning units of this course should basically follow the course for STEM students. Because of this constraint, determinant is taught at the end of the spring semester, just after learning Gaussian elimination. However, the context for determinant proposed in Table 2 is a Markov chain, population dynamics, or PCA that is related to diagonalization taught in the fall semester. Thus, we could not provide an effective context for determinant. Currently, the context for learning determinant is given by Cramer's formula for solving a system of linear equations with a square coefficient matrix and by the formula for the inverse matrix. This context is far from applications in statistics and data science. In the AM style, mathematical/computational methods are better to be taught "on a need-to-know basis" (National Research Council, 2003). Thus, it may be better to teach determinant with eigenvalues in the fall semester. However, this is beyond the discretion of the lecturer, so it is necessary to advocate the university administration to separate the course from the course for STEM students.

Time constraint

Regarding the linear algebra course presented in the previous section, each semester course is a 2-credit course that meets for 15 weeks of 90-minute classes with 1 week for the exam period. Es-

pecially in the fall semester, even a traditional MM-style linear algebra course is packed with content and there is little time to spare. Incorporating learning contexts for the AM style will require using class time, which means that the content will have to be reduced compared to normal MM style lectures. If the curriculum constraint does not allow the content to be reduced, it will not be possible to sufficiently incorporate learning contexts for the AM style due to time constraints. The inability to effectively incorporate learning contexts for inner product spaces (orthogonal diagonalization), which we mentioned earlier, is also influenced by time constraints.

Textbook issue

As mentioned in the previous section, we could not find a suitable textbook for this new linear algebra course that was written in the AM style. Therefore, we decided to use an elementary linear algebra textbook as the textbook for the course and designed the lessons in the AM style by adding the context presented in the previous section to the content of the textbook. As a result, the class is very different from the textbook. Some students expressed dissatisfaction with this in the free-form written comments in the survey at the end of the 2024 academic year: “The lessons themselves were easy to understand, but there were things that were not explained in the textbook or differed from the textbook, and sometimes I couldn’t understand well even after looking at the textbook and was a little confused.” The best way to improve this situation is to write a textbook for the course. Writing a textbook is one of the next tasks.

Concluding Remarks

In this article, we have discussed mathematics teaching from the perspective of incorporating applications. Based on the discussion of “applicationism” by Barquero et al. (2011), we classified mathematics teaching styles into three styles: MM style (teaching mathematics only in mathematical contexts), MA style (teaching mathematics first and applications later), and AM style (starting from application situations to teach mathematics). We focused on the third style, the AM style, whose essence is to “teach mathematics through the context of applications”. Among the three examples of the AM style presented in the fourth section, the first and the third examples showed that this style is promising in teaching mathematics to non-mathematics students (non-STEM students in the first example and life science students in the third example). From the observation of the three examples, we presented a guideline for designing a course in the AM style in the fifth section. There is another important point to emphasize. In all three examples of the AM style presented in the fourth section, non-mathematicians were involved in the development of the courses and textbooks. Many mathematicians are not familiar with the applications of mathematics outside mathematics. For the successful development of an AM-style course, it is very important for mathematicians and mathematics educators to collaborate with specialists from outside mathematics, as has been emphasized in many studies (e.g., Kumar & Jalkio, 1999).

In the sixth and seventh sections, we reported on the design and implementation of an AM-style linear algebra course for non-STEM students linked to statistics. We examined the effectiveness of the course design and the learning contexts with application problems in the context of statistics and data science, and pointed out course design issues, such as context development, curriculum constraints, time constraints, and textbooks. Many challenges remain in the development of this linear algebra course. The linear algebra course presented in this article is in its fourth year and is still being improved. We would like to address the issues presented in the previous section one by one in our future practice.

Finally, I would like to mention two general issues regarding AM-style education in universities.

The first is that in the AM style, it is necessary to customize the application context to meet the needs of the students. In the case of UCLA's reform of calculus for life science students, this approach has been very successful. The linear algebra course for non-STEM students reported in this article has been basically successful, although further improvements are needed. However, is it possible to develop application contexts for students in other fields? If linear algebra or calculus is necessary for students' learning in their major, there should be a specific need for these mathematical concepts of calculus and linear algebra, so it seems possible to identify and address this need. However, the context from this need may be too difficult for first-year students learning linear algebra or calculus. As seen in the part of the linear algebra course reported in this article where PCA did not function well as a learning context, this could be a challenge in course development.

The second issue is that mathematics teachers who teach AM-style courses require knowledge and skills that are different from those required for traditional MM-style mathematics courses. In the case of the calculus reform at UCLA, mathematicians who were not involved in the development of the course also are teaching the course, which can be considered a success in this regard. In the author's case, the course (Mathematical Literacy 1, 2) presented as the first example of the fourth section succeeded in creating a system so that several mathematicians could provide consistent instruction by preparing detailed lesson plans (Kawazoe, 2022). However, the situation remains that only a limited number of mathematicians are available to teach the course. The linear algebra course reported in this article has not yet reached a stage where it can be taught by any other teachers. While the development of lesson plans, as in the case of the Mathematical Literacy course, will be necessary, we must also consider that the qualifications required to teach AM-style courses appear to be different from those required to teach traditional mathematics courses. It will be necessary to consider teacher training programs to cultivate mathematicians capable of teaching AM-style courses. While there is previous research on the knowledge required for teaching mathematics through application (Doerr, 2007), it focuses on pre-service secondary school teachers and does not address teacher training. Research targeting the university faculty is needed.

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Invited Lecture 2

**COMPUTATIONAL THINKING AS A CATALYST FOR CHANGE:
REFLECTING ON WHAT, HOW, AND FOR WHOM WE TEACH
MATHEMATICS IN THE DIGITAL AGE**

Oi-Lam Ng

**INTEGRATING AI AND DIGITAL TECHNOLOGIES IN STEM+I
MATHEMATICS EDUCATION: OPPORTUNITIES AND CHALLENGES**

Ho Kyoung Ko

**EMPOWERING STUDENTS AS MATHEMATICIANS THROUGH
COMMUNITY-BASED MATHEMATICAL MODELING**

Jennifer M. Suh

**TEACHING MATHEMATICS THROUGH PROBLEM POSING:
A SYSTEMATIC REVIEW OF PROBLEM-POSING-BASED LEARNING
RESEARCH**

Jaepil Han & Jinfa Cai



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



COMPUTATIONAL THINKING AS A CATALYST FOR CHANGE: REFLECTING ON WHAT, HOW, AND FOR WHOM WE TEACH MATHEMATICS IN THE DIGITAL AGE

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This lecture explores three key dimensions of integrating computational thinking (CT) into mathematics education. (1) “What”: I begin by stating that, in an increasingly digital world, manual calculations and procedural tasks are increasingly automated, making problem-solving, reasoning, and digital/AI literacy more critical than rote learning or procedural fluency. In turn, what is valued in mathematics education must shift toward prioritizing thinking skills, as demonstrated in my empirical research on CT integration in mathematics education. (2) “How”: I present task designs from research that position mathematics learning as analyzing problems, designing computational solutions, and interpreting algorithmic outputs. This shift requires moving beyond paper-and-pencil methods to computational environments where students engage in programming-based explorations. (3) “For Whom”: I end with discussing equitable access to computationally enriched mathematics education. Through knowledge transfer initiatives, I highlight strategies to support learners with low digital literacy, fostering inclusive participation in CT-based learning. By framing CT as both a catalyst for change upon mathematics curriculum and instruction, this lecture informs future research on preparing students to engage critically and computationally in mathematical practice in an AI-driven world.

Keywords: computational thinking; digital literacy; problem-solving; technology-enhanced learning; equity

Introduction: Computational Thinking as a Catalyst for Change

As an essential thinking skill that can be applied to diverse problem-solving situations, it is suggested that CT should be acquired by everyone starting from early years of education in a similar manner as early literacy and mathematics (Wing, 2006). Accordingly, the global movement of cultivating CT has permeated formal education, with the purpose of not only preparing the next generations to accommodate the digital and AI-powered society that is yet to come, but also providing a powerful way for school children to express themselves through design, experimentation, and problem-solving in computational environments (Resnick et al., 2009). In addition, the importance of CT has been underscored by research suggesting positive effects on school children’s cognitive development (e.g., executive function, mathematical concepts) and 21st century skills (e.g., creativity, collaboration, communication, critical thinking; Bers et al., 2022; Sung et al., 2023; Weng et al., 2022).

To date, there is a lack of consensus on the definitions and frameworks of CT due to its comprehensive and interdisciplinary nature across age groups in and out of school settings (Aho, 2012; diSessa, 2018; Papert, 1980; Wing, 2006). Notable models in the field include the framework proposed by Wing’s (2006) CT definition, focusing on abstraction, analysis, and automation (3A’s) as essential problem-solving and thinking skills; Aho’s (2012) definition of CT through the lens of computation theory, stating that CT involves “the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms” (p. 832); as well as Brennan and Resnick’s (2012) work on children’s block-based programming activities, consisting of three facets of CT: foundational computational concepts (e.g., sequences, loops, parallelism),

computational practices that build on conceptual knowledge (e.g., being incremental and iterative, testing and debugging), and the perspective of using computing as a means of communication (e.g., expressing, connecting, and questioning). Other commonly adopted definitions of CT are used by researchers in the field, as encompassing a set of competencies, including problem decomposition, pattern recognition, algorithmic thinking, and abstraction (Angeli & Giannakos, 2020; Cansu & Cansu, 2019; Shute et al., 2017). Despite the inconsistencies in the definitions of CT components and their varying focuses and contextualization, CT can be viewed generally as a competence across problem-solving activities (Grover & Pea, 2013), with specific interdisciplinary connections to mathematics (Lodi & Martin, 2021), where notions like ‘decomposition’, ‘algorithms,’ ‘loops,’ and ‘testing and debugging,’ can be fruitfully interpreted from both programming and mathematics perspectives (Benton et al., 2017; Miller, 2019; Ng et al., 2023; Weintrop et al., 2016). In this paper, my main thesis is to unpack such an interplay among CT, mathematics and problem-solving; in doing so, I highlight the role of CT as a possible catalyst for change for current mathematics curriculum and instruction.

An interplay among CT, mathematics, and problem-solving: Example in early childhood context

To illustrate, I offer an example of CT as a catalyst for change, in the context of mathematical problem-solving within a mircroworld, TouchCounts (TC). The “change” described in this example refers to how the child, Susan, approached the problem in increasingly sophisticated manner, demonstrating growth in mathematical thinking. The episode also shows how early CT can be developed and manifested without a programming tool, but rather, within a touchscreen application. This suggests, more generally, that CT is a generic skill across different tool-based learning and problem-solving environments, and therefore, from an equity perspective, it is crucial to provide access and support development of CT inclusively from an early age (more details in the section “For Whom We Teach Mathematics in the Digital Age”).

The episode illustrated here involves a task that the adult researcher (myself) presented to two first-graders about devising efficient strategies to determine the total number of blocks of the rectangular model without resorting to laborious one-by-one counting while utilizing TC (Figure 1). It contrasted with what took place moments earlier, when Susan responded “I don’t know” upon my question of “how do you know when to stop (with TC)?”

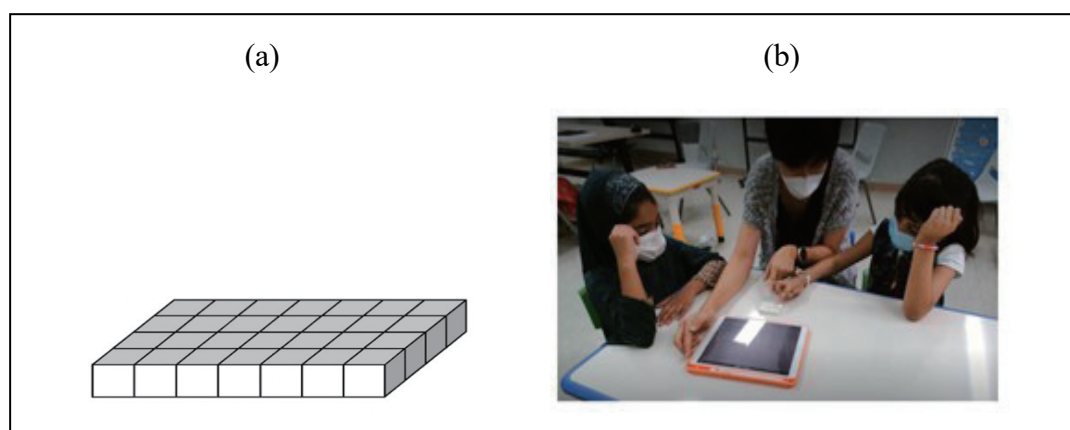


Figure 1a. An example of the rectangular model task given to the participants. Figure 1b. A participant devising a strategy of finding the number of unit blocks in a random rectangular model without one-by-one ‘counting’ (Yeung & Ng, 2024)

While solving the task of finding the total number of unit blocks in a 5x7 arrangement, the following interaction unfolded among Susan, myself, and TC (adapted from Yeung & Ng, 2024):

Susan:

One (*Putting five fingers on TC*)

TC:

Five!

Susan:

Two (*Putting five fingers on TC*)

TC:

Ten!

Susan:

Three (*Putting five fingers on TC*)

TC:

Fifteen!

...

Susan:

(*Iterating the process of putting five fingers on TC*) Seven

TC:

Thirty-Five

Susan:

(Total number of blocks is) Thirty-Five!

...

Researcher:

Can you explain what you did?

Susan:

I counted there are one (*pointing to the unit of blocks*), two (*pointing to the unit of blocks*), three (*pointing to the unit of blocks*), four (*pointing to the unit of blocks*), five (*pointing to the unit of blocks*), six (*pointing to the unit of blocks*), seven (*pointing to the unit of blocks*), so I tapped seven times (on the screen). [Figure 2]

Susan:

I didn't count (them one by one), but I counted this, this, and this (*referring to the three steps of the gesture scheme*).

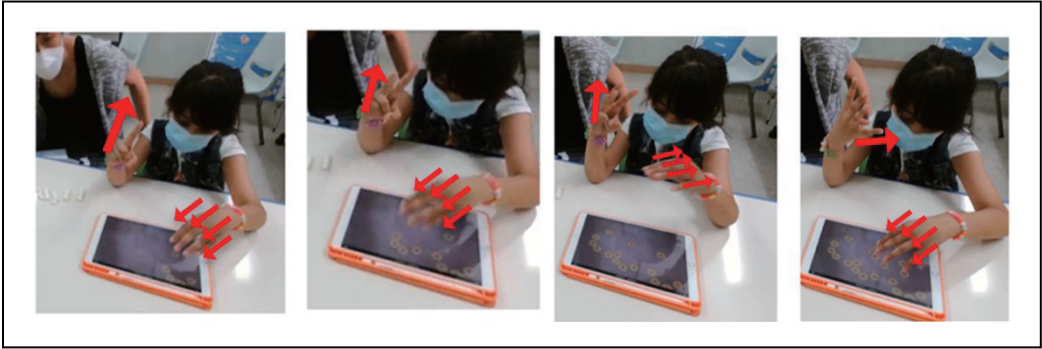


Figure 2. An illustration of child-tablet interaction and the corresponding computational concepts being enacted (Yeung & Ng, 2024).

As shown in the above, Susan taps her left hand with five fingers against TC while TC chants “5.” Following the tapping, Susan uses her right hand to signify “1,” indicating that the iteration has occurred once. In what follows, Susan taps TC with her right hand again, adding an additional finger with each iteration. When examining closely both hands of Susan at this moment, they worked together to enact algorithmic thinking: as she interacted with the touchscreen using her left hand,

the number stored in TC remained constant, signifying the gesture of “5” as a constant input for computation. Meanwhile, each tap of “5” performed an iteration in the computation in the sense of adding five more to the sum; this showcases how she leveraged her left hand to enact the algorithmic process. Concurrently, her right hand enumerated the iteration count, extending one finger at a time to track the number of iterated units. Unlike the left hand, her right hand remained stationary, and the number of fingers changed during the course of problem solving, underscoring its function as a variable (in a programming sense, variable as container) with each incremental count.

As she repeated this process, her body adapted to and synchronized the rhythm of left-hand tapping and right-hand extending, culminating when the value in her right hand matched the desired number of taps. This rhythmic and coordinated two-handed action enacted the iteration construct of loops—a fundamental CT concept in various programming languages.

This episode exemplifies how core elements of CT—such as iteration, variables, and algorithmic processes—can emerge within young children’s problem-solving activities, even in the absence of formal programming tools. Susan’s embodied interaction with the touchscreen environment not only demonstrates her algorithmic reasoning but also illustrates how CT and mathematical thinking are deeply intertwined in her approach to solving the problem. The authentic problem context and developmentally appropriate task prompted her to identify patterns, abstract numerical relationships, and develop a solution systematically, all of which are foundational to both mathematics and CT. This interplay underscores how CT can serve as a lens through which mathematical ideas are explored, articulated, and extended. In this way, CT enriches the learning of mathematics by shifting the focus from procedural execution to strategic, conceptual engagement with problems. Importantly, the example also challenges the misconception that CT requires access to complex coding environments; instead, it shows that CT development can be meaningfully addressed in early childhood context.

Research Background and Framework

Over four decades ago, Seymour Papert pioneered an educational approach along with the Logo programming language (Papert, 1980; Noss & Hoyles, 1992), which envisioned a paradigm shift of school mathematics—from manipulating with numbers, symbols and formulae by paper-and-pencil to developing powerful ideas about the world by using computers as a flexible, playful material. Papert’s vision of CT focuses on mathematics learning through constructionist practices, or “learning-by-making” (Papert & Harel, 1991), which highlights the context of actively exploring and experimenting through artefact construction, and of constructing domain knowledge from the activities. The contemporary notion of CT is explicitly described by Wing (2006) as “solving problems, designing systems, and understanding human behavior, by drawing on the concepts fundamental to computer science” (p. 33). Such definition of CT has been popularized for its application across disciplines as comprising a set of problem-solving competence and attitudes developed with or without computers. Since then, a wide range of notions associated with CT competence has emerged, including a set of syntactic, semantic, schematic knowledge (e.g. loops, conditionals, etc.) and strategic problem-solving skills (e.g. abstraction, pattern recognition, etc.). As Lye and Koh (2014) explain, an explicit CT framework is necessary and appropriate for understanding how K-12 students approach programming, particularly for visual programming languages like Scratch, which shares similar features and are widely adopted in K-12 education contexts. However, conceptual frameworks that focus on the role of CT in school disciplinary learning, particularly in mathematics (Benton et al., 2017; Kallia et al., 2021; Weintrop et al., 2016; Ye et al., 2023), remains underexplored in extant research. Future research in this area will contribute toward research and practice

of CT-mathematics integration by understanding how computational and (school) mathematical practices converge.

In Ye et al. (2023), we examined how CT has been integrated into K–12 mathematics education through recent empirical studies. Our review focused on the types of CT-based mathematics instruction implemented and the forms of student learning documented in the literature. Across the studies surveyed, a wide array of CT components was identified. From these, we synthesized seven key CT concepts that commonly emerge in programming-rich mathematical contexts: variables, conditionals, loops, sequences, events, subroutines, and operators. Furthermore, we identified a hierarchy of CT practices linked to programming-based mathematical activities, including decomposition, abstraction, pattern recognition, algorithmic thinking, automation, modularization, modelling, and simulation (Ye et al., 2023). Importantly, each CT concept and practice can be defined strictly from a programming perspective as well as can be interpreted from a mathematical perspective. While these CT concepts and practices are non-exhaustive, our systematic review serves to initiate understanding of how CT concepts and practices are contextualized in programming-rich mathematical activities through the lens of empirical studies.

To advance the above findings theoretically, we extended the interpretation of CT from programming and mathematical perspectives by incorporating the concept of a boundary object from sociology (cf. Star & Griesemer, 1989). Specifically, we drew on the idea of a boundary object as something that is flexible enough to be adapted to the specific needs and constraints of different groups, while at the same time maintaining a coherent identity across contexts. In common use, it tends to remain loosely structured, but within individual settings, it can become more specifically and strongly structured (Star & Griesemer, 1989). Building on this understanding, we conceptualized computational thinking (CT) as a boundary process that activates varying interpretations while retaining sufficient structure to facilitate mutual understanding between the mathematics and computer science communities in problem-solving contexts (Ng et al., 2023). This is because CT represents a shared cognitive approach used by both mathematics and computer programming learners during problem-solving activities. As such, CT is adaptable enough to support problem solving across mathematical and programming contexts, while also being robust enough to sustain common problem-solving heuristics across both STEM domains (Leung, 2020). By “common problem-solving heuristics,” we refer to aspects of CT that are recognizable and consistent across both the mathematics and computer programming communities. Our empirical investigation in Ng et al. (2023) demonstrates the value of framing CT as a boundary process—a shared template for problem-solving and reasoning in both mathematics and programming. In the next section (analysis of Task 2), I provide an example of the utility of this framing, identifying decomposition, pattern recognition, and loops as boundary processes in a mathematical activity of polygon making in Scratch. More generally, I adopt such a perspective of CT in the design and implementation of CT-based mathematical activities, not limited to conventional programming.

What Mathematics (and How) Do We Teach in the Digital Age

Indeed, researchers have long devoted effort in realizing the potential of CT in mathematics classrooms. Such effort has been motivated by critiques for a shift away from teaching mathematics as “initiation of decontextualized knowledge” (Sutherland, 1991, p. 45): “Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas?” (Sutherland, 1991, p. 46). For Sutherland and the Logo team, algebraic ideas could be addressed in more intuitive and concrete ways with Logo programming. Their work has demonstrated students’ growth in understanding variables, working with variable-dependent ex-

pressions, and using variables to express a general method upon sustained engagement in Logo. Like the work of the Logo team (Hoyles & Sutherland, 1989), in the past few years, I have been interested in designing problem-solving situations in which we may take advantage of programming (in my research, mainly Scratch) to assist problem-solving. Under such context, we have generated fruitful findings in understanding the unique interplay among mathematics, CT, and problem solving; two such examples will be shared in this paper (see also, Ng, 2025, for a relevant critique and discussion).

As mentioned already, CT is not only drawn upon in programming practices, but also a competence when solving problems, having conceptual connections with mathematical thinking and learning. Given this unique interplay among CT, mathematics, and problem-solving—and the lack of emphasis on CT in the current mathematics curriculum and instruction—it follows that we may fruitfully consider CT as a catalyst for change, through rethinking (and reforming) “what mathematics curricula do we teach, and how do we teach it, in today’s digital/AI age.”

To elaborate, I introduce the following two tasks which have been designed and adopted in my research projects.

Task 1: The Number Machine

In this task, students will identify the patterns in the given number machines, design their own number machines and their “inverses” on Scratch, and finally use and make guesses about their peers’ number machines. In mathematical terms, the number machines can be thought of as functions which take a certain number input to be mapped onto a number output based on certain rules. Therefore, the learning objectives of the lesson included grasping the concept of functions, their inverses, and variables, but we intended to achieve these outcomes implicitly, as our target participants were upper primary students. Following Polya’s problem-solving cycle, we guided students in developing their programming solutions in Scratch. First, to understand the problem (Stage 1), students were to engage initially with four different number machines developed by the teacher initially (Figure 1a). These number machines differed by the kinds of arithmetic operations on the input, e.g. $f(x)=x+7$, $f(x)=3x$. In this initial stage, they experimented and conjectured with different input values, testing if their results validated their prediction of how the number machines were defined. At the same time, they were also actively engaging in *pattern recognition*, an important competence across CT, problem-solving, and mathematical perspectives.



Figure 3a. A student’s number machine, $f(x) = 5x$;
Figure 3b. Another student’s number machine, $f(x) = 75 + x$

Next, they were prompted to make their own number machines and “reverse” machines (which are essentially inverse of $f(x)$) in Scratch. To do so, they needed to devise a plan (Stage 2), in other words, to algorithmicize a procedure that would perform the way their intended number machines would. In terms of teacher scaffolding, we suggested the students to explain how their number machines work step-by-step. Specifically, by observing the given number machines, they set out to program their machines such that the cartoon character would prompt for an input and would display the corresponding output three times in sequence; then, it would further prompt the user to guess the output given if the input was “12”. Finally, it would respond “Correct” or “Incorrect. Try again.” depending on the user’s response.

The students carried out their plans (Stage 3) to create their target function machines in Scratch. They utilized the programming blocks provided in Scratch, including “ask”, “say”, “if... then... else...”, and “repeat” to complete the task. Moreover, they looked back (Stage 4) and checked their machines by running their machines and identified any errors. Their final task was to program the inverse version of the original number machines, which takes the output and processes it to return to the original input according to the machine they have made.

In our research, we experienced with tasking students as early as fifth-grade with this task. The task can be characterized as “low-floor, high-ceiling”, as students from diverse mathematical and programming backgrounds can enter the problem, by playing with the teacher-made number machines initially. Then, we observed that the students wrote down their input and output values on paper to look for the inherent patterns or rules that governed the machines. We highlight that, here, CT serves as a catalyst for change because under this programming environment, CT transformed students’ algebra learning from solving routine problems of “one-step equations in one unknown” to conceiving number patterns with variable-dependent expressions. By contrast, many students in conventional mathematics curricula have difficulties grasping the meaning of variables (Ely & Adams, 2012). The task is also “high-ceiling” in that students were free to make their own number machines not limited to using the four basic arithmetic operations. They are also encouraged to think about what makes a function inverse, which is an intuitively algebraic question.

When designing this task, we were not necessarily targeting students to acquire some learning outcomes in mathematics; rather, we were interested in fostering CT as a generic problem-solving skill (Wing, 2006), through which mathematics thinking serves as the means, not an end in itself (Ng, 2024). This is because CT is an essential skill that we need to encourage in the real world; it is about recognizing where mathematics is applicable, interpreting practical problems into mathematical problems, as well as solving and evaluating the problem solutions in a computational context (Wolfram, 2022). By taking a “typical” algebra/function task in the conventional curriculum and redesign it from a CT perspective with Scratch, I note that the learning outcome of “solving one-step equations in one unknown” is no longer required, nor the terminologies of variables (“ x ”), expressions, and equations. Rather, students engage in the problem computationally and are tasked by an experimental approach that helps to form heuristic knowledge in mathematical problem-solving (Grover & Pea, 2013; Feurzeig et al., 2011). Meanwhile, CT elements such as pattern recognition, algorithms, programming variables, and conditionals, are drawn upon necessarily in their solutions.

Task 2: Drawing Regular Polygons

In this task, students will explore the properties and draw various regular polygons, leading toward the ultimate goal of drawing a regular 10-sided polygon in Scratch. Following Polya’s problem-solving cycle, we helped students understand the task (Stage 1) by showing a demonstration of a Scratch program which drew a 10-sided polygon. During the demonstration, the students counted

the number of sides of the polygon to figure out their task of drawing a regular 10-sided polygon. When devising a plan (Stage 2), we prompted them to first draw a simpler case (e.g., triangle, square), and then to generalize a method to draw a regular n -sided polygon with larger number of sides (e.g., pentagon, hexagon). To draw a simpler case in Scratch, the students observed the properties of the polygon's exterior angles, such as 90° for square and 60° for equilateral triangle.

The students then went on implementing their plan (Stage 3) with or without realizing how the exterior angles were related to the number of sides of a regular polygon. Having been familiarized with Scratch blocks of "move X step," "turn X degrees," and the "Pen up/down" functions, they mainly performed trial-and-error to complete the sub-task of drawing triangles and squares. From there, they needed to devise a plan for drawing other regular polygons, which required them to explore the relationship between the exterior angle (i.e. $360^\circ/n$, which is also the angle turned by the cat character each time upon each forward movement) and the number of sides of the regular polygon, n . To facilitate this, we prepared to show a dynamic image in which the direction of the cat before and after the turn was marked to support visualization that the sum of all turns should yield 360° (Figure 2).

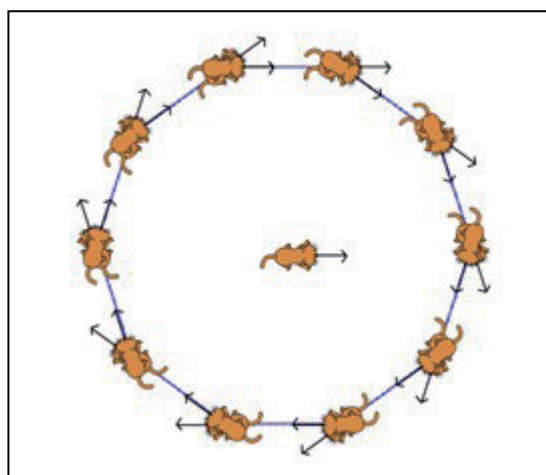


Figure 4. A dynamic image shown to students to support visualization of the sum of exterior angle of polygon

Finally, the students looked back and checked whether they had completed drawing the required shape (Stage 4). As an extension of their learning, they were invited to make a program that draws any n -sided regular polygon upon the user inputting the value of n into the program. Characterized by the constructionist environment, some students were interested in carrying out their own exploration: A student started creating artwork using her 10-sided polygons (Figure 3a), while another explored the way to draw a "seeming circle," stating that the smaller the degree of each turn, the more the polygon would look like a circle (Figure 3b). Throughout this process, the students consistently looked for patterns, conjectured, tested and debugged about the angle turned by the cat and its relationship with the number of sides of the intended regular polygon made.

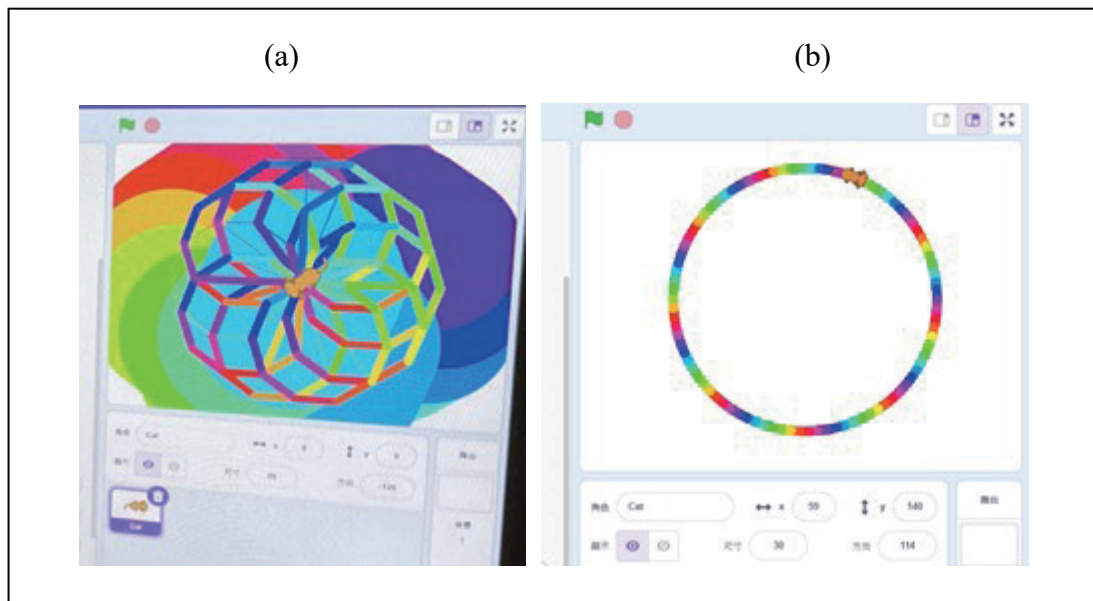


Figure 5a-b. Students' artefacts created upon the 10-sided polygon task

Reflecting on the tasks

Astonishingly, more than 30 years after the inception of Logo programming, Papert's (1991) prediction that computational practices would revolutionize (mathematics) education did not take place; to date, teaching mathematics, in particular algebra, remains to be largely an "initiation of decontextualized knowledge" (Sutherland, 1991, p. 45). In response, my research has attempted to envision a computationally enriched mathematics education (Ng & Cui, 2021), for instance, such that algebraic ideas are implicitly developed (as in Task 1) and geometrical concepts are explored alongside with programming learning (as in Task 2). These tasks take advantage of the tool affordances of programming, concretizing abstract mathematical ideas while supporting generalizations of various relationships and properties. While not intending to suggest that the computational approach should replace the conventional approach in school mathematics, I do argue that the computational approach deserves more attention and consideration than currently received, for they serve as a good context for learning CT and computer-assisted problem-solving which have been largely under-represented in school mathematics. This aligns with viewing CT as a catalyst for change upon the current mathematics curricula ("What") and instruction ("How") by considering the affordances of CT-based mathematics learning.

Furthermore, this consideration also prompts us to rethink how to better organise the K-12 mathematics curriculum, given that computer-assisted problem-solving can facilitate diverse CT and mathematics learning outcomes and trajectories. From Task 1, there is a shift in mathematical thinking from solving an equation by paper-and-pencil to engaging in a programming activity of conceiving, constructing, and conjecturing about two sets of numbers, such as $\{1, 2, 3, \dots\}$ and $\{3, 6, 9, \dots\}$. As shown in Task 2, various CT elements such as decomposition, pattern recognition, and loops can be identified as boundary processes within the geometry construction (Ng et al., 2023). Taking decomposition as an example, while its critical features vary between the mathematical and programming perspectives, there are also shared aspects. From a mathematical perspective, decomposition involves breaking a regular polygon into n congruent parts (one-dimensional components and angles). In contrast, from a programming perspective, decomposition entails representing the

figure as a sequence of defined movements, such as turning angles and advancing forward. Even though mathematics learners and programming learners may possess different prior knowledge regarding the figure and its sub-components, the process of devising an algorithm to construct the figure fosters meaningful communication about decomposition across both perspectives (Ng et al., 2023). Gradually, the programming activity facilitated the student's exploration of the sum of exterior angle property, which evidenced their mathematics learning even if it required only basic knowledge about angles. This change of geometrical knowledge supports what I mean by CT as a catalyst for change. Therefore, when viewed as a boundary process, CT embraces multiplicity in interpretation and mediation between what count as mathematics learning and programming learning.

For Whom We Teach Mathematics in the Digital Age

In this part of the paper, I would like to tackle the question, “for whom” we teach mathematics in this current digital/AI age. Thus far, I have discussed that CT is not the exclusive thinking skill of computer scientists, but a form of digital literacy and higher-order thinking skill that everyone should possess in problem solving, wherein mathematical thinking and learning can productively emerge. As such, it has been widely suggested that every citizen should have access to CT education. In 2022, the Hong Kong Education Bureau (EDB) initiated the “Senior Secondary ICT Curriculum Update”, focusing on fostering students’ CT and programming skills across compulsory education. Despite continued efforts by EDB to support quality CT education, there remains a significant and research-evidenced gap in students’ digital literacy and problem-solving competence (Reichert et al., 2020), especially in disadvantaged communities with limited access to technology. According to the Hong Kong Poverty Report 2020, the number of children living in poverty is as high as 274,900, with a child poverty rate of 27% (HKSAR, 2020). Previous research has shown that children from underprivileged communities have lower academic achievement (OECD, 2019), particularly in terms of CT skills (Karpiński et al., 2023). Thus, there exist urgent needs for more inclusive and sustainable CT education to provide equitable opportunities for youth from disadvantaged backgrounds to develop CT and prepare for the future.

In the episode with Susan’s interaction with TouchCounts, I have shown that CT can be meaningfully addressed in early childhood context through accessible, well-designed tasks that support young learners’ reasoning and problem-solving. Recognizing and fostering such opportunities is essential to promoting equitable and developmentally appropriate CT integration across diverse educational settings. In the following, I share some empirical findings from a Knowledge Transfer Project aiming to address, leverage, and respond to equitable practices in fostering digital citizenship and STEM education.

The Knowledge Transfer Project

The rest of this paper details the impact of a Knowledge Transfer Project, titled “Empowering Youth from Underprivileged Community: Enhancing Programming Skills and Mathematical Knowledge for STEM Teaching and Learning” on students’ conception of learning during programming-based mathematical activities. The project is informed by my previous research findings which generated evidence-based practices for teaching mathematics in computing-enhanced environments (e.g. Cui & Ng, 2021; Ng et al., 2021; Ng & Cui, 2021). These studies delineate the big picture of how using computers might fundamentally change STEM learning, with an emphasis on mathematical problem-solving. Besides, I utilized the concept of responsive teaching to develop teacher expertise in technology-rich environments while attending to students’ diverse needs (Ng & Chan, 2021). These research efforts have shed light on multiple ways to enhance school children’s CT and mathematics

learning, as well as teachers' professional knowledge in technology-rich pedagogy. Ultimately, the research project aims to address, leverage, and respond to equitable practices in fostering digital citizenship and STEM education.

The said Knowledge Transfer project aims to improve CT and mathematics education through teacher and student empowerment. Through partnerships with schools and community organizations, the project team organized a series of activities as follows:

1. Offering a free-of-charge, 10- to 15-hour course (entitled “編程數學家” or “I’m a Programming Mathematician”) to primary and secondary schools in Hong Kong.
2. Providing practical teaching experiences for pre-service mathematics teachers undertaking a BEd teacher education programme.
3. Conducting professional development workshops to share experiences with in-service teachers.

From January 2024 to date, we partnered with 5 local schools and 1 non-profit community organization to provide a free-of-charge course (entitled “I’m a Programming Mathematician”) in an extra-curricular after-school setting (Figure 4). Our partnered organizations respectively serve underprivileged youths in various regions in Hong Kong. This approach of directly collaborating with local communities aims to fill the educational gap caused by resource disparities. Moreover, we worked closely with local mathematics teachers' associations to develop teacher expertise in Hong Kong, focusing on how to (and why) integrate CT in mathematics curriculum and instruction. This was achieved by organizing workshops to meet the professional development needs of local schoolteachers.



Figure 6. Snapshots of the course, “I’m a Programming Mathematician”

In this paper, I share the project impact in terms of students' conception of learning in the course “I’m a Programming Mathematician”, at two of our partnered primary schools. To evaluate students' conceptions of learning in programming and in mathematics classrooms, we conducted interviews with participants selected from the primary schools served. The selection process ensured representativeness, and the sample consisted of eight students (two girls and six boys). Interviews were conducted in pairs, aiming to create a comfortable environment for students to share their thoughts openly.

Each interview session lasted approximately 30 to 40 minutes. As a key component of the evaluation, students were asked to draw three pictures in the following order: “anything that made the biggest impression on you,” “an image of yourself engaging in the programming class” and “an image of yourself engaging in your regular mathematics classroom.” Follow-up questions were used to encourage students to elaborate on their drawings, providing rich qualitative data for analysis. This method offered valuable insights into students’ perceptions, emotions, and experiences in both learning environments.

The analysis of interview data revealed stark contrasts between students’ conceptions of learning in Scratch programming classes compared to traditional mathematics classrooms. The findings are categorized into themes that highlight differences in teaching methodologies, student engagement, and the overall learning environment in response to the drawing task and semi-structured interviews.

Agency and active learning

In Scratch programming lessons, students experienced a heightened sense of agency and empowerment in their learning process. They described these classes as student-centered, emphasizing active participation and self-directed exploration. A predominant sentiment from the drawings is an abundance of happy feelings in the programming class, with many smiley faces drawn (Figure 5). One student remarked, “My mind is full of play,” reflecting the enjoyment and engagement they felt while learning. Students often perceived the activities as “playing” rather than “studying,” which contributed to their enthusiasm and intrinsic motivation. Another student noted, “I feel so energetic in Scratch class,” underscoring the stark difference in emotional engagement compared to mathematics lessons.

In contrast, mathematics lessons were perceived by students as mainly teacher-centered, with a recurring sentiment that they followed a predictable format: One interviewee succinctly summarized their mathematics classes as “teachers endlessly talk, talk, talk, and students do, do, do.” Perhaps because of this, students’ drawings reflected themselves in the mathematics classroom as unenthusiastic (Figure 5). The said structure and focus on direct instruction impacted students’ engagement, such as feeling disinterested and disconnected compared to the excitement they felt in the programming lessons.

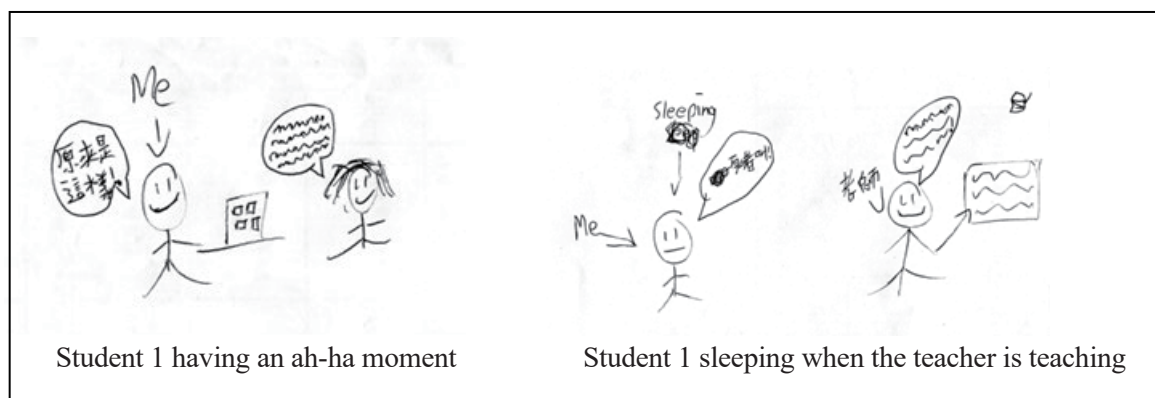


Figure 7. Drawings of Student 1 in a programming lesson (left) and in the mathematics classroom (right)

Open-ended nature of learning with multiple pathways to success

Students’ perceptions of problem-solving differed significantly between Scratch programming and mathematics lessons (Figure 6). Students appreciated the open-ended nature of programming in Scratch lessons, where multiple solutions were possible. One student commented, “There’s also right and wrong in programming, but when I get something wrong, I might actually discover something new.” The emphasis on testing one’s artefact as a final step to programming was undertaken as a step of discovery and problem-solving. Another student commented that, with programming, “possible versions of the answer are limitless”. This contrasted sharply with mathematics classrooms, where students felt there is only one right answer. Students expressed a disdain for checking answers in mathematics lesson, “only to discover their answers were wrong”, which they felt discouraged their learning.

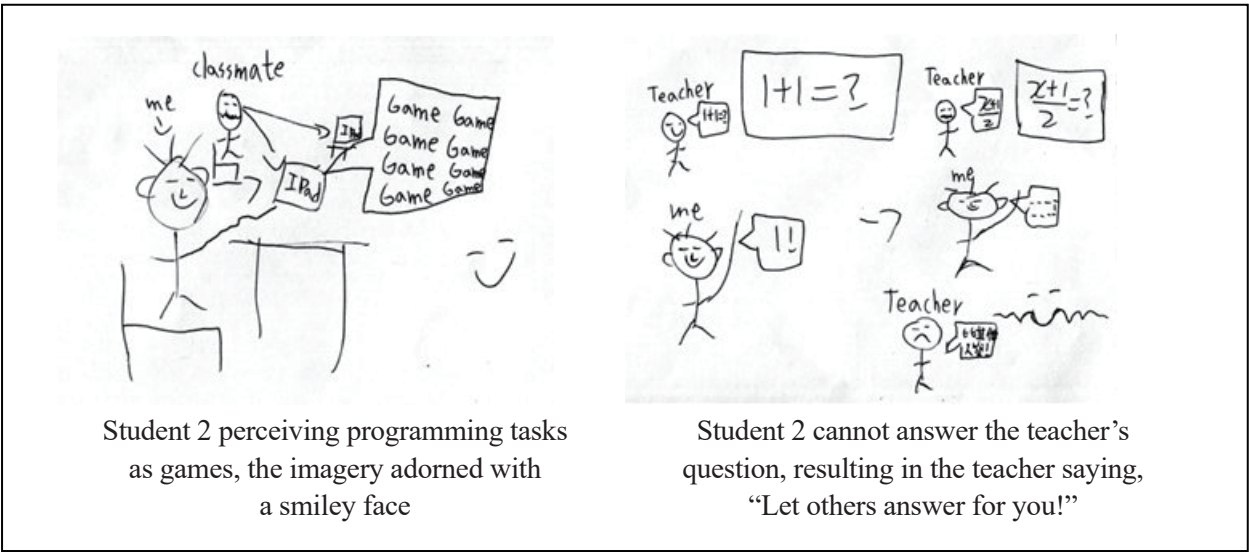


Figure 8. Drawings of Student 2 in a programming lesson (left) and in the mathematics lesson (right)

During the interview, students were invited to name one thing that had particularly impressed them. An interviewee drew the below, remarking with a sense of pride that “even the teacher did not know of my way” (Figure 7), suggesting not only freedom, but also a kind of victory, that students could chart their own pathways to success without the need to resort to that one correct, well-trodden path.

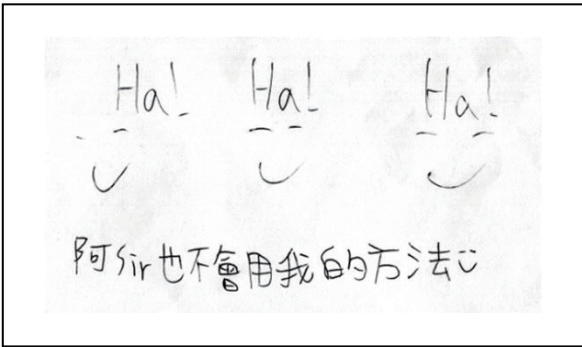


Figure 9. “Even the teacher did not know of my way”

Individually rich and meaningful learning

Scratch programming lessons offered students a personalized learning experience. They engaged in creative projects that allowed them to see the tangible results of their efforts. Many students took pride in their creations, describing them in detail during interviews. When asked to recall what they did during the programming lesson, students were able to recount in concrete details the artefacts they had made, for example, a story about a “roast goose” or a “鬼滅之刃” guessing game. These artefacts were not only visually rich but also carried personal meanings for the students, making the learning process enjoyable. One student commented, “Seeing my work alone makes me happy, also a happy experience to put together the building blocks.”

The students appreciated the tangible outcomes of their work. With the ability to create a game or an interactive project, the students extended their enthusiasm to learn and make things that are personally meaningful. For example, some shared their aspirations to create video games or applications that can help others relieve stress, or to create games that can trick fellow students.

In mathematics lesson, however, students described their learning as abstract and disconnected from their personal interests. The focus was on solving exercises with predetermined answers, leaving little room for individual creativity. Students noted that their learning often felt repetitive and lacked meaningful application beyond completing tasks. One student remarked, “Mathematics class is just exercises,” emphasizing the lack of variety. Their satisfaction came from “knowing how to solve a problem”, and through continued practices, “being able to solve the problem quicker”.

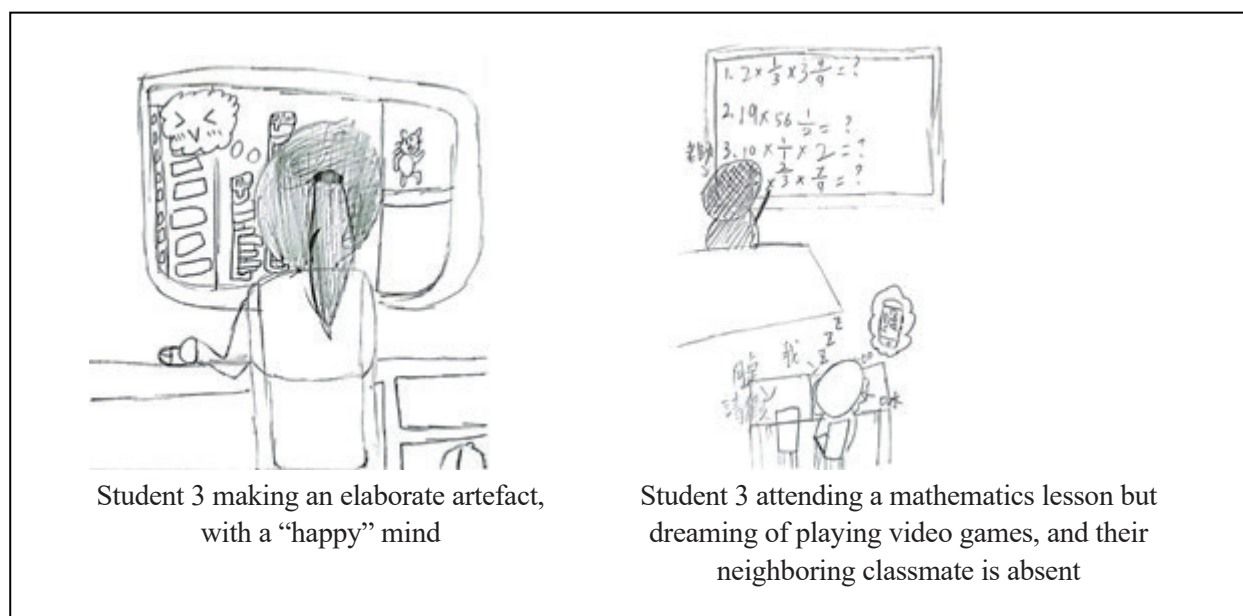


Figure 10. Drawings of Student 3 in a programming lesson (left) and in a mathematics lesson (right)

Meanwhile, Scratch lessons fostered a collaborative environment where students shared their projects and received feedback from peers and teachers. This sense of community enhanced their motivation, with students enjoying opportunities to showcase their artefacts. One student remarked, “Other students may come sometimes to look,” to check out on their progresses, emphasizing the value of recognition and shared learning. In contrast, mathematics lessons were characterized by

individual effort and a lack of interactive opportunities. Knowledge was primarily transmitted from teacher to student, leaving little room for collective problem-solving.

Discussion and Conclusion: Are We Ready for Change?

In this paper, I demonstrated the impact and potential of interpreting CT and integrating computationally enriched activities into mathematics education, particularly in fostering active learning, exploration, and personal experiences in the mathematics classroom. The activities bridged CT with mathematics, affording students to develop generic problem-solving skills as well as their abstraction and pattern recognition competencies in a student-centered environment. Notably, the findings revealed that students often did not perceive these activities as “doing math,” but as an enjoyable and empowering process of self-learning, discovery and creation. One student shared with excitement, that they wish “all mathematics classes one day become programming classes”. While the sentiment could be exaggerated, it hints at a path to better leverage new technology to inspire and enhance learners’ enthusiasm.

The contrast with traditional, teacher-centered mathematics lessons underscores the need for pedagogical shifts towards problem-based and computationally enriched learning environments for mathematics. While this paper focuses on the positive impacts of programming-based approaches, the project also recognizes that systemic inequalities, particularly in disadvantaged communities, pose significant barriers to equitable participation. More work is needed to address equitable practices to empower students from diverse backgrounds and communities. In this light, we observe that the participating students felt overwhelmingly positive about their learning experiences, expressing a sense of agency and satisfaction upon the lessons. By empowering both students and teachers through the Knowledge Transfer Project, we hope to cultivate a new generation of learners equipped with the skills and confidence to thrive in a digital world.

In sum, this paper has argued that CT serves as a productive lens for re-envisioning mathematics education in the digital age. By examining what mathematical knowledge is valued, how it is taught, and for whom it is accessible, we have shown that CT-oriented pedagogies can meaningfully support students’ development of mathematical reasoning, problem-solving, and digital literacy. The examples discussed, spanning early childhood settings to programming-integrated classrooms, demonstrate that CT is not confined to coding, but rather constitutes a set of cognitive and epistemic practices that intersect with core mathematical processes such as abstraction, generalization, and algorithmic thinking. While the Knowledge Transfer Project provides encouraging evidence of the potential for CT to foster student engagement and agency, it also underscores the imperative to address structural inequities that may limit access to such opportunities. As mathematics education continues to evolve in response to technological and societal shifts, future research must attend to the design of inclusive, developmentally appropriate, and context-sensitive approaches to integrating CT. Doing so will not only enhance the relevance of mathematics education but also support the cultivation of computationally literate and critically engaged learners prepared to participate meaningfully in a forthcoming, AI-driven society.

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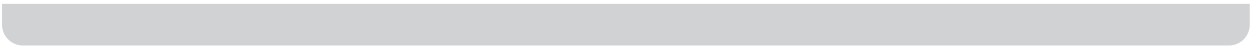
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INTEGRATING AI AND DIGITAL TECHNOLOGIES IN STEM+I MATHEMATICS EDUCATION: OPPORTUNITIES AND CHALLENGES

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With the rapid advancement of artificial intelligence (AI) and digital technologies, mathematics education is experiencing a significant paradigm shift. This paper explores the integration of AI and digital tools into STEM+I (Science, Technology, Engineering, Mathematics + Information) mathematics education and also considers the use of AI Digital Textbooks (AIDT) as part of this transformation. Drawing on empirical research and classroom implementations, the study presents a comprehensive review of instructional models, teaching resources, and teacher training programs that embed AI technologies into mathematics education. Key focus areas include AI-math integrated lessons using natural language processing (NLP) and optimization, student-led chatbot development projects, and personalized learning supported by AI-based platforms such as Sussam. In addition, the paper discusses the development of dialogue-based AI chatbots for conceptual inquiry and outlines models for enhancing mathematics teachers' AI competencies through inquiry-based and career-integrated training. Findings highlight the potential of AI to enhance conceptual understanding, self-directed learning, and learner engagement, while also underscoring the need for systemic innovation in curriculum, pedagogy, and teacher preparation. This paper offers practical insights and future directions for building a sustainable, learner-centered ecosystem for AI-enhanced STEM+I mathematics education.

Keywords: Artificial Intelligence, AI Digital Textbook (AIDT), STEM+I Education, Mathematics Education, Teacher Training, Personalized Learning

Introduction

The Fourth Industrial Revolution and the rapid development of artificial intelligence (AI) technologies are accelerating a paradigm shift in mathematics education. As interest grows in integrating AI literacy, digital literacy, and mathematical thinking, increasing efforts are being made to design curricula and teaching methods that promote an understanding of AI principles. The 2015 revised national curriculum in Korea reinforced software (SW) education (Ministry of Education, 2015), while the 2022 revision systematically incorporated core Fourth Industrial Revolution technologies such as AI, data science, and big data, aiming to cultivate students' problem-solving and creative thinking abilities (Ministry of Education, 2022).

In particular, understanding key AI technologies such as machine learning requires strong mathematical foundations in areas including probability, statistics, calculus, and linear algebra (Ko, 2020). Accordingly, the Ministry of Education has implemented policies to strengthen AI and big data education in mathematics (Ministry of Education, 2020a) and introduced a new high school elective course titled "AI Mathematics" (Ministry of Education, 2020b).

In this context, mathematics education research has increasingly embraced AI integration, with multiple case studies reporting improvements in students' mathematical thinking, problem-solving ability, self-efficacy, and interest through Python-based coding activities (Kim & Ko, 2020).

This paper explores strategies for integrating AI and digital technologies into STEM+I mathematics education. It presents specific examples, such as AI-math integrated learning activities on natural language processing (NLP) and optimization, and student-led chatbot development projects. Fur-

thermore, as AI technologies become deeply embedded in society, economy, and industry, classroom teaching and learning must evolve to utilize AI more effectively. The AI Digital Textbook (AIDT) plays a central role in supporting personalized learning tailored to students' levels and individual characteristics, thereby facilitating self-directed learning environments (KERIS, 2023).

The author has investigated the use of AI-based learning platforms (e.g., Sussam) to support personalized instruction and has developed teacher training programs for the effective use of AI Digital Textbooks. By presenting these real-world cases, this paper seeks to provide practical implications for the innovation of mathematics education.

Development and Implementation of AI-Math Integrated Teaching and Learning Materials

NLP-Based Teaching Materials

Kim and Ko (2024) developed a series of teaching and learning materials that integrate natural language processing (NLP) techniques into mathematics education, leveraging Python-based activities in conjunction with ChatGPT. This interdisciplinary approach aimed to deepen students' conceptual understanding of mathematics while simultaneously enhancing their digital literacy competencies.

The instructional design consisted of six class sessions, in addition to a preparatory session, each linking core NLP methods with specific mathematical concepts. In the initial sessions, students engaged with unstructured text data by employing the Bag-of-Words (BoW) model, transforming natural language sentences into vector representations. They then applied mathematical tools such as Euclidean distance and cosine similarity to compute semantic relationships between texts. These activities were strategically aligned with students' prior knowledge of vectors and inner product operations, thus facilitating the practical application of abstract mathematical concepts to real-world problems.

Subsequent sessions centered on classification and prediction tasks, wherein students constructed basic N-gram language models to calculate the conditional probabilities of word sequences. These activities served to illustrate the foundational mechanisms underlying large language models, such as ChatGPT, particularly with respect to generating contextually appropriate responses. Students implemented these models in Python, experimented with varying N-gram sizes, and explored smoothing techniques.

[Figure 1] illustrates one such activity from the sixth session, in which students generated multiple N-gram codes using Python and evaluated how changes in the N-gram window size affected the model's output. By modifying and executing diverse forms of code—such as unigram, bigram, and trigram models—students were able to explore how predictive performance and computational logic change depending on the structure of word sequence modeling. This hands-on exercise provided a concrete context for applying conditional probability, enhancing both statistical reasoning and algorithmic thinking.

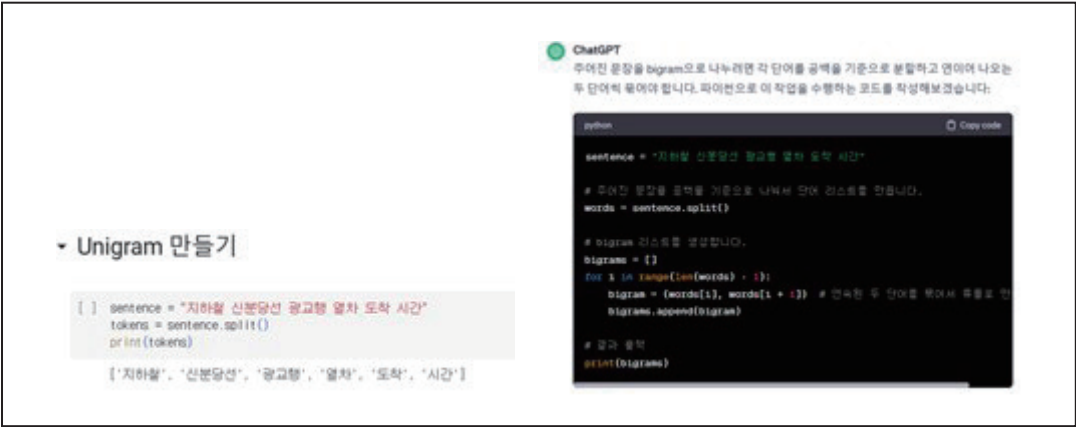


Figure 1. Example of creating various N-gram codes using ChatGPT’s programming functions (Kim & Ko, 2024, p.496)

A distinctive feature of this study was the utilization of ChatGPT not merely as a code generator, but as an active partner in the co-construction of knowledge. Students engaged in iterative, reflective dialogue with ChatGPT to refine their code and deepen conceptual understanding. When confronted with coding errors or logical inconsistencies, students consulted ChatGPT for diagnostic feedback, revised their scripts accordingly, and validated improvements through successive testing cycles. This approach fostered both computational thinking and metacognitive strategy development.

The overall pedagogical framework emphasized mathematical reasoning and AI literacy, integrating topics such as text representation, similarity measures, probabilistic modeling, and algorithmic thinking. Moreover, the lessons were designed to support key curricular goals, including data visualization, logical inference, and the application of mathematics in authentic contexts. In sum, the integration of NLP-based tasks supported by ChatGPT offered an effective interdisciplinary platform for fostering mathematical understanding and AI-related competencies. The findings of this study suggest that, when embedded within a coherent instructional framework, large language models such as ChatGPT can function not only as instructional tools but also as interactive agents that support learner engagement and conceptual development.

Optimization-Focused Lessons

The study by Lee and Ko (2024) developed a set of teaching and learning materials for the “Optimization” unit in high school mathematics, emphasizing intuitive understanding of Mean Squared Error (MSE) and Gradient Descent through practical Python programming activities supported by ChatGPT. This instructional design aimed to deepen students’ grasp of mathematical modeling by engaging them in hands-on, iterative tasks using authentic data and code-based simulations.

In the early sessions, students explored how to fit trendlines to datasets by minimizing the MSE, which measures the discrepancy between predicted and actual values. They began by using simplified datasets—such as height and weight or mock academic scores—to understand how different prediction functions yield different errors. Students used ChatGPT to generate Python code that computed MSE for a given function, then tested various candidate functions by modifying code parameters and interpreting the resulting MSE values. These explorations helped students intuitively grasp that the function with the lowest MSE best represents the data trend.

Building on this, the lessons progressed to the formulation of regression functions in the form $y = ax + b$, and students investigated how adjusting coefficients a and b affects the MSE. Through repeated trials and interactive coding, they began to understand the need for a systematic method—gradient descent—to find the optimal parameters that minimize error. ChatGPT assisted not only in code generation but also in debugging, interpretation, and code refinement, allowing students to improve both their computational thinking and self-directed learning skills.

In the final sessions, students created and visualized Gradient Descent algorithms using ChatGPT-generated code. The program allowed them to adjust key hyperparameters such as learning rate, initial values, and iteration count. By simulating these changes, students observed how the algorithm approaches the function's minimum and how improper parameter settings might hinder convergence. For learners unfamiliar with calculus, the instructional team explained the concept of gradient as a slope and visualized the idea of descending a hill to reach the lowest point—offering a concrete metaphor for understanding abstract optimization theory.

The pedagogical value of this approach was reinforced through student-generated outputs. As shown in Figure 2, the left panel depicts the Python-generated graph of various prediction lines and their corresponding MSE values, while the right panel shows a student's completed activity sheet that outlines their logic in selecting the optimal prediction function and calculating associated errors. These artifacts not only demonstrate content mastery but also reflect students' ability to engage in mathematical reasoning, visualization, and interpretation of results using AI-supported tools.

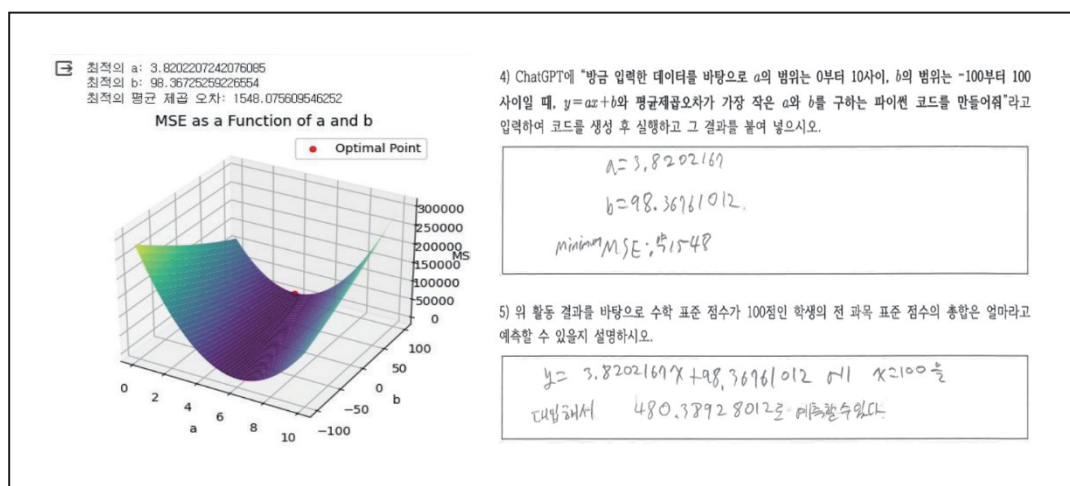


Figure 2. Example of the results of the program generating the mean square error of a minority (left) and the student activity result sheet for predicting optimization (right)(Lee & Ko, 2024, p.479)

In summary, the study illustrates that integrating ChatGPT and Python into AI mathematics education enables students to actively construct knowledge through inquiry-based, data-driven exploration. The focus on optimization, contextualized through MSE and Gradient Descent, provided a meaningful connection between traditional mathematics and cutting-edge AI applications. The success of this model supports its applicability to broader educational contexts and suggests promising directions for AI-integrated math instruction.

Student-Led Chatbot Projects

The study by Yoo and Ko (2023), “The Effect of Math Project Learning Using Chatbot on Artificial Intelligence Literacy,” examined how integrating chatbot development into project-based mathematics learning could enhance high school students’ artificial intelligence (AI) literacy. This research is situated in the broader educational context of preparing students for the AI era, aligning with the goals outlined in the Korean Ministry of Education’s push for computational thinking and digital literacy education.

Participants included 41 high school students from different grade levels who voluntarily engaged in an after-school program spanning six sessions. The project was built on a learning progression that started with understanding the fundamentals of AI—such as data, information, and procedural knowledge—and culminated in designing and deploying a functional chatbot using the Kakao i Open Builder platform. Students were given autonomy to choose project topics, allowing for diverse themes such as school guidance systems, scientific concepts, ethical debates (e.g., GMOs), and practical tools like study assistants and hobby-related chatbots.

The instructional design was deeply grounded in project-based learning (PBL), promoting self-directed learning, collaboration, and iterative refinement. Students not only implemented their procedural logic into chatbot structures (using scenario blocks, intents, and parameters), but also simulated real-world applications where logical reasoning, problem decomposition, and user interaction modeling were critical. The hands-on experience of deploying a chatbot to respond intelligently to user queries offered students a functional understanding of AI concepts like natural language processing, rule-based systems, and algorithmic thinking.

Quantitative findings revealed a statistically significant increase in students’ AI literacy scores, with particularly strong gains among female students. For example, in second-year students where a comparison group was included, the experimental group outperformed their peers with notable effect sizes. Moreover, qualitative analysis of student-created chatbots highlighted their successful application of knowledge structuring and scenario design, with examples ranging from Rubik’s cube solving to DNA education bots.

One of the most compelling contributions of the study is its demonstration that chatbot development can serve as a dual-purpose educational tool: both as an AI-enriched medium for delivering content and as a pedagogical platform that allows learners to embody AI principles through design. Furthermore, students reported high satisfaction with the project, citing the novelty, autonomy, and practical relevance of the tasks.

In sum, this study confirms that integrating chatbot development into mathematics project-based learning fosters AI literacy and essential 21st-century skills. It highlights the potential of AI tools not only to support learning but to transform students into creators of intelligent systems. Future research may explore extending this model to younger age groups or more advanced AI capabilities such as intent classification or reinforcement learning integration.

Application of AI-Based Mathematics Learning Platforms and Tools

The Sussam Platform

The study “A Case Study on Middle School Classes Utilizing the Math Learning Application Sussam”(Yook, Huh, & Ko, 2024) introduced Sussam, a digital mathematics learning task management platform developed to facilitate personalized and self-directed learning. The platform is composed of a teacher-facing web-based management system, “Sussam-MathTeacher,” and a student-facing mobile application. It enables real-time interaction between teachers and students and promotes a more adaptive, data-driven instructional environment (Figure 3).



Figure 3. ‘Sussam’ developed as a mathematics learning task management platform (Yook, Huh, & Ko, 2024, p.15)

Teachers use the platform to assign problem sets tailored to individual student needs. Through this system, educators can monitor each student’s progress in real-time, evaluate their achievement of core mathematics competencies (such as problem-solving, reasoning, and information processing), and provide personalized feedback based on both cognitive and affective data. The teacher interface offers granular control over task assignment, including problem difficulty, topic alignment with national curriculum standards, and mode of delivery (AI-recommended or teacher-selected).

Students engage with assigned tasks through the app, which features an intuitive interface that highlights upcoming or pending tasks. As they solve problems, they are encouraged to document their reasoning using an integrated digital notebook, promoting process-based evaluation and self-reflection. After submission, students complete a self-assessment that covers key affective dimensions—such as satisfaction, interest, effort, and confidence—which the teacher can later review to better understand each learner’s disposition and needs.

One of the platform’s strengths lies in its ability to generate rich learning analytics. For example, the system captures response accuracy, time on task, and submission behavior, generating visual reports that help teachers adapt instruction accordingly. Instructors can review aggregated data by class, task, or individual student, enabling differentiated instruction and more strategic remediation efforts.

Moreover, students benefit from the application’s ability to support remediation through targeted review and revision. They can revisit incorrectly solved problems, reflect on errors, and receive scaffolding for improvement. Additional features like the “discussion room” allow students to communicate with teachers or peers for clarification and support without being constrained by time or location.

The Sussam platform not only streamlines task distribution and feedback but also empowers students to take ownership of their learning. Its emphasis on formative assessment, coupled with interactive features such as note-taking, self-evaluation, and visual analytics, aligns with broader educational goals of fostering metacognitive awareness and self-directed learning habits. The case study highlighted measurable gains in both cognitive skills (e.g., problem-solving) and non-cognitive outcomes (e.g., confidence and learning satisfaction), demonstrating Sussam’s effectiveness as a blended learning tool in middle school mathematics classrooms.

Development of a Dialogue-Based Chatbot for Conceptual Inquiry

In the study “Development of a Dialogue-Based Chatbot Model for Middle School Mathematics Concept Inquiry,”Ko (2024) introduced “Suribot,”an interactive, dialogue-based educational chatbot specifically designed to guide students in exploring key mathematical concepts in a self-directed and inquiry-based manner. The chatbot was developed as part of a broader effort to create a student-centered digital learning environment that integrates multimodal data and AI-driven personalization.

The primary goal of Suribot is to support students’ conceptual understanding of core mathematical domains—such as rational numbers, functions, and geometry—by providing step-by-step explanations, scaffolded hints, and conceptually connected follow-up questions. Unlike conventional Q&A-type bots, Suribot employs a scenario-based dialogue structurethat enables dynamic and adaptive interactions based on students’ responses, levels of understanding, and inquiry intentions.

[Figure 4]presents a scenario design structure in which the chatbot not only answers students’ immediate questions but also naturally continues the dialogue by offering related follow-up questions or learning content links. This design encourages continuous exploration and deepens conceptual understanding through sustained engagement.

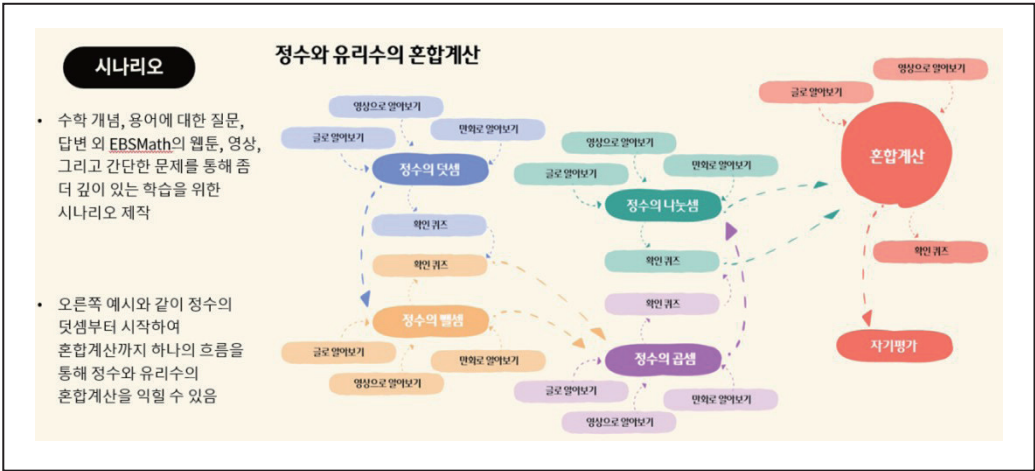


Figure 4. ‘Suribot Scenario Design Structure Example

In addition to providing textual responses, Suribot integrates multimodal learning supports, including graphs, diagrams, formula renderings, and interactive elements. The system is designed to accommodate diverse learning styles by incorporating both visual and verbal explanations. For example, if a student asks about “the meaning of slope in a linear function,” Suribot may respond with a verbal definition, a graph showing how slope affects the line, and a follow-up inquiry such as “What happens when the slope becomes negative?”—thus fostering both conceptual clarity and critical thinking.

Another key feature of Suribot is its support for linking conceptual learning with actionable tasks or practice opportunities. As illustrated in [Figure 5], after learners explore a given concept, they are encouraged to solve a set of simple problems or continue learning through linked content that builds upon the prior topic. This function enhances the quality of learning and facilitates learner reflection, ensuring that the chatbot conversation becomes a vehicle for both cognitive and metacognitive engagement.



Figure 5. After learners learn a specific concept, they can solve simple problems or continue learning related concepts.

Suribot also functions as an intelligent guide that adapts to individual learner progress. The chatbot provides immediate clarification when students show misconceptions and encourages reflective dialogue when students respond correctly, thereby scaffolding both mathematical thinking and learning autonomy. Its backend includes tools for customizing content flows and collecting learner interaction data, enabling teachers to monitor student progress and adjust instruction accordingly.

In essence, Suribot represents a new class of educational chatbots that transcend static fact delivery to serve as interactive, intelligent learning companions. By structuring learning through personalized dialogue, affective interaction, and multimodal content linkage, Suribot fosters deep conceptual inquiry and supports students in becoming reflective, self-directed learners.

Enhancing Mathematics Teachers' AI Competency

Teacher Training for AIDT Utilization

The study “Development of a Teacher Training Model for Enhancing AI Digital Textbook Utilization Competency” (Ko & Ahn, 2025) proposed a practice-based teacher training model aimed at cultivating educators’ capacity to effectively utilize AI Digital Textbooks (AIDT). Recognizing the pedagogical shift toward personalized, self-directed, and data-informed learning environments, the study emphasized the importance of enhancing teachers’ digital and AI literacy to meet the evolving demands of 21st-century education. The model was developed by integrating three theoretical foundations: the TPACK framework, Cognitive Load Theory, and Learning Analytics. These frame-

works were operationalized within the structure of *Forschendes Lernen* (inquiry-based learning), guiding teachers through a cyclical process of inquiry, application, and reflection.

The training program was structured into three phases. In the Inquiry Foundation phase, teachers analyzed national AIDT policy, studied international trends, and formulated their own practical research questions grounded in school contexts. They engaged with foundational theories related to AI pedagogy and reviewed data collection tools aligned with AI-based diagnostics and personalized feedback systems. In the Inquiry Method Development phase, teachers explored AI functions embedded in AIDT—such as automated scoring and dashboard-based learning analytics—through hands-on activities. They applied this knowledge to design personalized AI-supported lessons, reflecting the TPACK domains of technological, pedagogical, and content knowledge. Cognitive Load Theory principles were embedded throughout to minimize extraneous cognitive load and enhance germane processing.

In the Inquiry Execution and Reflection phase, participating teachers implemented their AI-based instructional designs in real or simulated classrooms, collected and analyzed student interaction data, and presented their findings to peers. This process was not only experiential but also dialogic, encouraging teachers to function as practitioner-researchers. Through structured peer feedback and guided reflection, they re-evaluated their instructional strategies and articulated improvements for future implementation. Such recursive engagement promoted a shift in teacher identity from knowledge transmitter to facilitator and co-learner, aligning with the goals of AI-enhanced learning environments.

The model further contributed to sustainable professional development by encouraging collaborative inquiry and adaptive use of AIDT technologies. It presented a departure from passive training formats, instead enabling teachers to gain firsthand experience with AI learning tools and pedagogical design grounded in data analytics. Ultimately, the study demonstrated that an integrated training framework grounded in theory and practice could empower educators to design and deliver more personalized, data-informed, and inquiry-oriented instruction using AIDT.

AI-Integrated Career Education Training

The study “Development of an AI-Integrated Education Program to Enhance Career Guidance Competency of High School Mathematics Teachers” (Ahn, 2025) designed a training program to help teachers guide students in exploring STEM careers by integrating AI knowledge. The program enhances teachers’ capabilities in three areas: (1) exploring STEM and AI careers through analysis of industry trends and interpretation of career assessments; (2) designing AI-math integrated lessons by linking math units (e.g., probability, statistics, functions) with AI algorithms; and (3) addressing AI ethics and social issues through modeling tasks and critical discussions. The program encourages teacher agency and supports teacher-researcher development through collaborative inquiry and reflection.

Conclusion

This paper has explored the opportunities and challenges in STEM+I mathematics education by integrating artificial intelligence (AI) and digital technologies. It presented practical cases in the development of instructional materials, implementation of AI-based platforms, and teacher competency enhancement programs. The key conclusions are as follows:

First, AI-integrated mathematics instruction demonstrates the potential to strengthen both conceptual understanding and real-life application. The organic integration of AI technologies—such as natural language processing (NLP), optimization, and conditional probability—with mathematical concepts provides students with new approaches to problem-solving and cognitive structuring, while also increasing learning interest and self-directed learning (Kim & Ko, 2024; Lee & Ko, 2024).

Second, AI-based mathematics learning platforms (e.g., Sussam) enable personalized diagnostics and feedback tailored to learners' levels, helping to bridge achievement gaps and positively impacting affective factors such as confidence, interest, and satisfaction. In particular, real-time learning analytics allow teachers to more precisely adjust instruction and support learners (Yook, Huh, & Ko, 2024).

Third, teachers' AI-related competencies must extend beyond basic tool usage to include practical inquiry skills, data interpretation, and the ability to guide ethical understanding. TPACK-based inquiry-oriented training programs and STEM career education models provide effective solutions. Teachers must continuously develop their professional identity as facilitators and instructional designers (Ahn, 2025).

Fourth, integrating AI and mathematics requires a fundamental restructuring of curriculum, pedagogy, and assessment. National-level systemic efforts are needed to develop AI-math integrated subjects, instructional guidelines, and teacher training systems (Noh, Ko, Kim, & Huh, 2023).

These recent studies emphasize the importance of personalized learning and complex problem-solving through the use of AI and big data (Yuk, Huh, & Ko, 2023), highlighting the increasing role of technological tools in mathematics education. With the 2022 revised curriculum emphasizing digital literacy, the national implementation of AI Digital Textbooks is accelerating. As a result, research on personalized inquiry learning and AI-driven instruction continues to expand (Sin & Ko, 2024), and there is a growing need for the Korean education community to actively develop instructional models that incorporate AI and cutting-edge technologies.

This evolution of technology-based mathematics education marks a critical turning point, calling for a fundamental shift from tool adoption to a restructuring of education grounded in learner-centered philosophies and creative problem-solving. Given the deep impact of AI on education, mathematics education must be restructured around these new principles (Ko & Ahn, 2025).

In conclusion, integrating AI and digital technologies into STEM+I mathematics education will serve as a foundational strategy for cultivating future-ready talents in a rapidly changing society. Achieving this goal requires a holistic innovation strategy that connects teachers, policy, platforms, and curriculum. This paper provides actionable insights and directions to support such a transformation.

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EMPOWERING STUDENTS AS MATHEMATICIANS THROUGH COMMUNITY-BASED MATHEMATICAL MODELING

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This paper explores community-based mathematical modeling (CBMM) as a means to engage students in rigorous mathematics while addressing real-world issues. Through co-designed lessons with teachers, students apply mathematical modeling to make informed decisions and take meaningful action on local challenges. Case studies illustrate how students collaborate on group-worthy tasks—ranging from environmental sustainability to inclusive design—using data, making assumptions, and validating models to solve local problems. The study investigates, (a) How do students engage in CBMM and group work? and (b) What facilitative moves support this engagement? Findings reveal that CBMM fosters collaborative modeling competencies, collective learning, and consensus building in students. The teacher’s intentional facilitative moves encouraged equitable participation, scaffolded thinking and empathetic reasoning.

Keywords: Community-based Mathematical Modeling, Socio-critical issues, Empathetic Reasoning

Introduction to Community-Based Math Modeling

Mathematical Modeling serves as a bridge between mathematics and real-world applications and a vehicle for addressing global challenges and fostering critical thinking (Stillman et al., 2015). In math modeling activities, students engage in key modeling practices posing problems; making assumptions; critically analyzing and interpreting models, reporting out solutions. Community-based mathematical modeling (CBMM) is an innovative approach that integrates social justice issues with mathematical education that emphasizes a component of taking action and leveraging students’ lived experiences to inform decisions and solutions. CBMM (Suh et al., 2023) has the potential to connect mathematics to real-world social issues, foster critical consciousness, and promote civic empathy among students. Rosa et al. (2023) discuss integrating cultural and community contexts into modeling, and highlights how students’ lived experiences can be used to create meaningful modeling tasks to support both mathematical understanding and citizenship education. Suh et al. (2023) defined community-based math modeling as an approach that immerses math within the context that involves community that builds awareness and has a component of taking action and is situated in the theoretical intersection of mathematical modeling and teaching mathematics for social justice (TMfSJ). TMfSJ is a pedagogical approach that encourages students to use mathematics to analyze and challenge societal injustices. Gutstein (2006) emphasizes that TMfSJ helps students “read and rewrite their world with mathematics, fostering a deeper understanding of power dynamics and social inequities. Felton-Koestler et al. (2017) argue that effective TMfSJ requires specialized knowledge that integrates sociopolitical and mathematical understanding. Suh et al. (2023) developed five design principles to guide the planning and implementation of CBMM tasks. These principles aim to address social justice issues while maintaining mathematical rigor and fostering student engagement:

1. **Situating Tasks in Authentic Local Community Issues:** CBMM tasks should be rooted in real-world problems that are relevant to students' lives and communities. This principle emphasizes the importance of connecting mathematical modeling to local social issues, such as food insecurity, to make learning meaningful and impactful.
2. **Exploring Data to Identify and Understand Social Issues:** Students should use mathematical tools to analyze data related to social justice issues. This principle encourages the use of statistics and proportional reasoning to uncover underlying factors contributing to problems.
3. **Using Rigorous Mathematics to Describe, Predict, Optimize, and Make Decisions:** CBMM tasks should involve complex mathematical processes that require students to make assumptions, identify variables, and validate models. This principle ensures that students engage in high-level mathematical thinking while addressing social justice issues.
4. **Encouraging Collaboration, Negotiation, and Justification:** Collaborative problem-solving is essential in CBMM. Students should work together to co-construct ideas, negotiate solutions, and justify their mathematical models. This principle promotes collective knowledge-building and civic discourse.

Taking Action on Community Issues: CBMM tasks should empower students to take action based on their mathematical findings. This principle highlights the importance of moving beyond awareness to implementing solutions that address social justice issues in their communities.

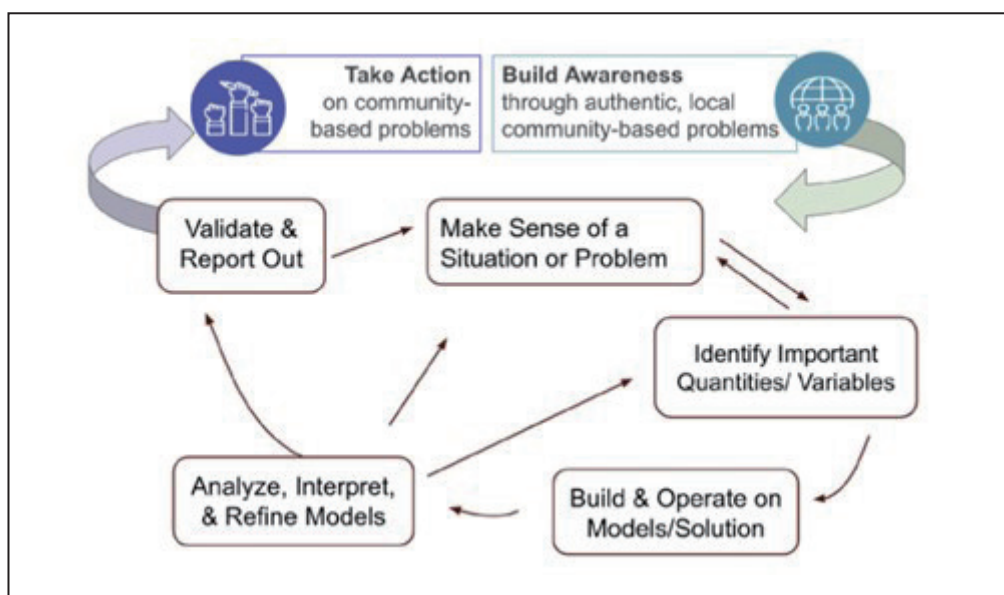


Figure 1. Community-based mathematical modeling with building awareness and taking action

Teaching CBMM can develop students' awareness of complex issues in our world and foster 21st-century skills such as critical thinking and creative problem-solving. CBMM tasks align with what Rosa and Orey (2015) refer to as socio-critical modeling, which situates mathematical modeling within cultural, economic, environmental, political, social, and natural contexts. The application of this socio-critical dimension allows mathematics to be seen as a dynamic and humanizing subject (Rosa & Orey, 2015; Gutiérrez, 2018). Turner et al. (2023) explore how community-focused

mathematical modeling in elementary classrooms can be a powerful tool for promoting equity and civic empathy. By engaging students in real-world, community-based problems, mathematical modeling tasks become more meaningful. These tasks allow students to explore social issues that affect both their own local community and others. They serve as mirrors, reflecting students' own lives and experiences, and as windows, offering views into the lives and perspectives of others. This approach helps make mathematics more humanizing, culturally relevant, and socially conscious, encouraging students to connect math to their lived experiences and broader social justice issues. Another way to immerse students in tasks that prepare them for the grand challenges of our world and develop 21st-century skills is by exploring areas outlined by UNICEF and the Sustainable Development Goals (SDGs). A study by Bulut and Borromeo Ferri (2025) explored how integrating Education for Sustainable Development (ESD) with mathematical modeling in a university seminar impacts pre-service primary teachers' understanding of sustainability and their ability to incorporate it into math education. The study found significant improvements in participants' knowledge of sustainability concepts, confidence in teaching ESD through modeling, and personal behaviors aligned with the SDGs. These findings highlight the effectiveness of modeling as a pedagogical tool for fostering critical thinking and real-world problem-solving in sustainability education.

Teaching CBMM can develop students' awareness of grand challenges in our present world and develop 21st century learners who can think critically and creatively as they problem solve. Community-based math modeling can focus on socio-ecological issues as well. Coles et al. (2024) emphasize the importance of integrating socio-ecological phenomena into mathematics education, advocating for a shift towards recognizing the interconnectedness of social and ecological issues. Mathematical modeling tasks can focus on socio-ecological issues in local areas, fostering a deeper understanding of the interconnectedness of social justice and environmental sustainability. This approach not only enriches mathematical learning but also empowers students to engage in meaningful activism and community-based problem-solving. For example, the first case study focuses on making bee hotels to support pollinators aligned with UN Sustainable Development Goal 15 (SDG 15): Life on Land. This goal aims to protect, restore, and promote sustainable use of terrestrial ecosystems, manage forests sustainably, combat desertification, and halt and reverse land degradation and biodiversity loss. More specifically, Supporting Pollinators addresses specific Target 15.5 to take urgent and significant action to reduce the degradation of natural habitats, halt the loss of biodiversity, and protect and prevent the extinction of threatened species and Target 15.9 to integrate ecosystem and biodiversity values into national and local planning, development processes. The second case study in this paper focuses on local issues that have global connections to the Education for Sustainable Development (ESD). The case study focused on designing an inclusive playground for all children aligned with UN Sustainable Development Goal 11 (SDG 11): Sustainable Cities and Communities. This goal aims to make cities and human settlements inclusive, safe, resilient, and sustainable. This focuses on Target 11.7 to provide universal access to safe, inclusive, and accessible, green and public spaces, particularly for women and children, older persons, and persons with disabilities. Inclusive playgrounds contribute to this goal by ensuring that all children, regardless of their abilities, have access to safe and engaging play environments.

The Importance of Collaborative Group Work and Discourse During Math Modeling

In community-based math modeling, student-to-student discourse is key to making decisions and brings creativity, and criticality to the solution process. Group work and mathematical discourse serve as powerful tools for collectively developing strategies (Stillman et al., 2020; Sjöblom et al., 2023), fostering a transformative learning environment where students and teachers co-construct

knowledge, challenge dominant societal narratives, and engage in rational dialogue essential for social transformation (Freire, 1970). Collaborative problem-solving in math supports civic reasoning and discourse (Lee et al., 2021) by encouraging students to explain their thinking, evaluate evidence, and engage in respectful dialogue. It helps them appreciate diverse perspectives and develop habits of listening and consensus-building. In mathematical modeling, group work creates rich opportunities for discourse, as students collaboratively build assumptions, propose solutions, and iteratively test and refine their ideas. In schools, modeling is primarily done in groups through collective reasoning and activity. To facilitate this in the project for which I am a member, the project team created a teacher move table for each of the modeling phases with questions and sentence stems to facilitate productive discussion that moves students to work with their classmates to make assumptions, build a model and use their solution to make decisions within a local issue they cared about eliciting the essential mathematical modeling practices. See sample questions and sentence stems related to making assumptions below.

Listing Quantities and Making Assumptions			
Elicit students' ideas about important quantities and assumptions, while supporting students to explain how each quantity/assumption relates to the problem posed			
Invite students to identify important quantities and explain how they relate to the problem posed <u>What this sounds like:</u> •What do you know that will help us with this question? •What do we need to know, or find out? •_____ said that we know how many students are in our class. How can that help us with our problem? •_____ said we need to find know _____. How can that help us?	Provide just in time information in response to key quantities that students identify <u>What this sounds like:</u> •You mentioned that you need to know _____. Here is a table with some information that might help. •_____ said that we need more information about _____. I did some research and this is what I found out. •_____ said we need to know how many students are in _____'s class. Let's find out.	Use targeted prompts and graphic organizers to help distinguish key components of the situation. <u>What this sounds like:</u> •You said _____ is important . Is that something we already know, or something to find out? •_____ said we need to know _____. Is this something we can find out, or do we need to make an assumption / best guess? •Let's use a chart to organize our ideas. This column is what we already know, and this one for decisions we need to make.	Ensure that students have opportunities to make (and revise) assumptions and decisions <u>What this sounds like:</u> •You get to make an assumption here. What seems reasonable? •_____ has an assumption will help us answer our question. Let's listen. •You get to make decisions with your group. It is ok to make different decisions. •_____ had an idea about a decision we need to make. •It's ok to revise your assumptions/ choices. What do you want to change?

Figure 2. Teacher moves to elicit discussion specific to listing quantities and making assumptions

Context for the Study and Methodology

The work presented in this paper stems from a four-year research and professional development project conducted across four diverse sites in the United States. The project focused on integrating established frameworks for mathematical modeling (COMAP & SIAM, 2016; Kaiser et al., 2011; Blum et al., 2007) with culturally responsive mathematics teaching practices (Zavala & Aguirre, 2024). Central to the project were collaborative design teams composed of researchers and teachers who co-designed mathematical modeling tasks. Professional development for participating educators included six comprehensive modules centered on culturally responsive mathematical modeling. The schools involved served culturally and linguistically diverse student populations, and the project emphasized the enactment of culturally responsive modeling tasks and routines across kindergarten through sixth grade.

Both case studies presented in this paper were drawn from lessons conducted by Ms. W, an experienced educator with seven years of teaching experience. Ms. W brought a natural passion for environmental issues into her teaching, which was reflected in the design and implementation of the community-based mathematical modeling (CBMM) tasks. Her third-grade classroom, located in a school in the Mid-Atlantic region of the United States, served a diverse group of 18 students. Of these, 16 were multilingual learners, and four received special education services, creating a richly inclusive and linguistically diverse learning environment. This context provided an ideal setting for exploring how CBMM can support equitable participation and foster mathematical thinking that is both rigorous and socially meaningful.

The guiding research questions focused on:

- a) In what ways do students engage in community-based mathematics modeling and group work?
- b) What are some facilitative moves by the teachers to support students in mathematical modeling and group work?

Data Sources and Analysis

To investigate these questions, a combination of video analysis and teacher self-assessment using the *Mathematical Modeling Lesson Analysis Tool (MMLAT)*. The *MMLAT* (see Table x) was employed as a formative assessment to support teacher reflection and document student engagement in key modeling practices. Following the enactment of modeling lessons, holistic scoring was conducted by teachers using the *MMLAT*, which included specific “lesson look-fors” across five key phases of the modeling process:

- *Problem Posing* – Making sense of real-world situations and posing mathematical modeling problems.
- *Identifying Important Quantities* – Recognizing relevant variables, making assumptions, and defining relationships.
- *Creating Models through Exploration and Representation* – Using mathematical tools and representations to develop and share models.
- *Interpreting and Revising Models* – Connecting models back to the real-world context and refining them as needed.
- *Reporting Out and Taking Action* – Communicating models and their implications to others.

Teachers were prompted to consider the extent of student participation in focal practices—such as problem posing, assumption-making, and model revision—using broad categories (e.g., a few students, half the students, most students). In addition to individual engagement, emphasis was placed on group work and group thinking, encouraging observation of how students collaborated and reasoned together. To further support instructional analysis, the Teacher Move Table was incorporated to identify facilitative strategies used to guide student thinking, promote equitable participation, and connect mathematical modeling to students’ lived experiences. This combined approach offered a practice-based lens for understanding both student engagement and teacher facilitation in culturally responsive modeling lessons.

Classroom lessons were video-recorded and to script teacher and student talk and actions. Detailed logs were generated to capture evidence of student participation and teacher facilitation across the phases of the modeling process. For each phase, the level of student participation was rated, and reflections were recorded on successes, challenges, and areas for growth. This tool supported both

formative reflection and systematic data collection on student engagement and teacher facilitation. To address RQ1, video evidence and MMLAT reflections were analyzed to identify patterns in student engagement across the five modeling phases. Attention was given to how participation was distributed and how students interacted with real-world contexts, mathematical tools, and peer collaboration. To address RQ2, teacher facilitation strategies that supported student engagement were examined. MMLAT reflections and video logs were reviewed to identify instructional moves aligned with culturally responsive mathematics teaching (CRMT), such as connecting to students' lived experiences, supporting assumption-making, and encouraging model revision. Analysis memos were used to document these practices and synthesize themes across lessons and grade levels. This integrated approach provided a comprehensive view of how mathematical modeling can foster equitable participation and support culturally responsive teaching in elementary mathematics classrooms.

Table 1. Mathematical Modeling Lesson Analysis Tool

Math Modeling Phases	Look Fors: Students ...	Reflection What went well? What was challenging? Areas for Growth?
1) Problem Posing- Making sense of real-world situations and posing MM problems	- Ask mathematical questions related to the situation - Use information and define the problem - Empathize/connect with the problem. - Bring in their experiences, funds of knowledge and cultural connections to the problem situation	
Rate the level of student participation while problem posing 1 ___ Few Students 2 ___ Half the Students 3 ___ Most Students		
2) Identify important quantities- Identify quantities, decisions and assumptions relevant to the problem	- Name and describe relationships between important quantities/variables - Recognize variables can change - Make reasonable assumptions, choices, and decisions - Assumptions stated are relevant to the situation and MM problem	
Rate student participation when identifying important quantities and assumptions for their model. 1 ___ Few Students 2 ___ Half the Students 3 ___ Most Students		
(3) Create models through exploration and representation - Mathematically explore possible models represented in various forms	- Reach a solution related to the problem posed - Use words, pictures, diagrams, tables and/or explanations to share a solution/model - Use math to demonstrate understanding of how the quantities are related - Correctly use mathematics in their model - Perform operation on the variables and relationships	
Rate student participation when exploring and representing their models. 1 ___ Few Students 2 ___ Half the Students 3 ___ Most Students		

(4) **Interpret and Revise Models-** Interpreting or revising models in relation to the real-world context

- Explain how their solution meets the situation
- Consider what if scenarios ...how the model might change with other variables
- Compare and contrast other solutions/ models to identify relationship among quantities
- Revise and refine the model if necessary

Rate student participation when interpreting and revising their models
1___ Few Students 2___ Half the Students 3___ Most Students

(5) **Report out and Take Action-** Communicate and explain the model to the audience. Share how the model might help others.

- Explain how their model works to solve the problems
- Identify limitations in their model
- Summarize and clearly communicate their model and solution
- Explain how their model can help someone else in a similar situation

Rate student participation when reporting out and taking action using their model.
1___ Few Students 2___ Half the Students 3___ Most Students

How did the lesson support equitable participation (multiple contributions, diverse ideas valued, high engagement)?

Findings

Case Study 1: Nesting Holes in the Bee Hotels

The bee hotel lesson was conducted over two class sessions, designed to engage third-grade students in environmental stewardship through the lens of ecosystem awareness and sustainability. Bee hotels mimic natural nesting habitats for solitary bees, offering a place to lay eggs, raise their young, and overwinter. The overarching goal was to help students understand the importance of bees and how they could contribute to conservation efforts by creating bee-friendly habitats, supporting local beekeepers, and raising awareness. On Day 1, students explored the causes of bee population decline and began constructing bee hotels as a hands-on conservation activity. On Day 2, the focus shifted to mathematical modeling, where students used their initial designs to estimate the number of paper rolls needed to build six bee hotels for their school garden.



Figure 3. Bee Hotel Modeling Task

The modeling task required students to identify important quantities and make assumptions (See MMLAT competencies 2) and apply estimation strategies to determine the dimensions of the bee hotels and the number of nesting holes each would contain. Working in groups, students engaged in collaborative problem-solving, applying math to a real-world environmental issue. Before the group got to work, students had an opportunity to work individually and come up with diverse solutions. This provided each and every student to engage in the cognitive demand of the task and have ownership in their mathematical thinking. Once they had some ideas on paper, the group engaged in “group-think” math talk



Figure 4. Teacher takes the paper rolls rolled by students and places them into the can

The teacher used a facilitation move that we have coined as “If I know _____ then I could _____” to support students’ modeling competencies of determining important quantities and making assumptions. For example, “If I know I have 7 paper rolls then I can _____.” Teacher took the paper rolls rolled by students and placed them into the can (see Figure 4) and said,

Teacher: We’re going to fill this up with this size. Okay and you guys said that there’s less than half and if less than half is already seven, I am wondering if we want to revise our answers. You guys need to come up with one answer.

At that point, the student came with different answers and had to negotiate their thinking. There were ideas on the table that to fill the small can, they would have 15, 21 and 28 (see Figure 5). An excerpt from the Group think Math Talk illustrates how students build on each other’s ideas and validates each other’s thinking.

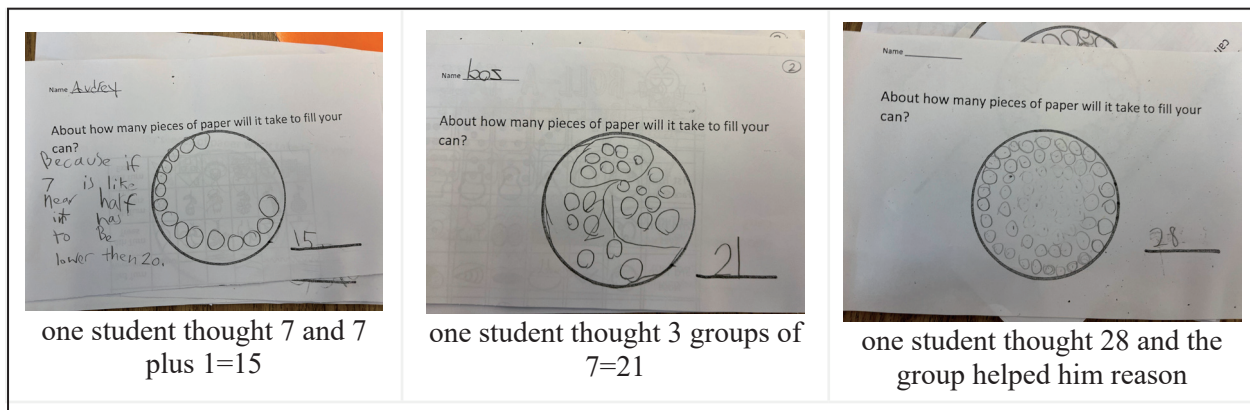


Figure 5. Students individual thinking on the number of paper rolls fit into a small can

Excerpt of group discussion describing relationships between important quantities/variables:

Teacher: (Put all the seven paper rolls in the can.) Based on what you see, your team will make an estimate.

Teacher: All right, what are you guys thinking?

James: I'm thinking [inaudible 00:00:09].

Selena: 15.

Teacher: 15. Okay, so let's look. How would you describe how much is being filled right now? But how would you describe it? Do you think it's about half?

Selena: About less than half.

Teacher: It's definitely less than half. I would agree. Okay, so if it's less than half. How many more of these do you think can fit into this can?

Selena: Okay.

Teacher: So if this is one right and you're telling me, you just said it's less than half.

Selena: I am going with Audrey

Teacher: Well Audrey's is fifteen. So Audrey's is basically saying that if this were to double. But you already told me it's not half.

Audrey: And then add one.

Teacher: And then add one. Okay. So we're going to fill this up with this size. Okay and you guys said that there's less than half and if less than half is already seven, I am wondering if we want to revise our answers. You guys need to come up with one answer. Okay. Well, let's talk as a group instead of just... So Yun I haven't heard your voice yet.

Yun: 21.

Teacher: Why do you think 21?

Yun: Because half is, no... One seven is... Yeah and then I add seven and then it's like half.

- Teacher:** No I love what you are saying there, yeah.
- Yun:** Seven's like half and then I add like another seven, I mean I don't...14.
- Audrey:** Do the math again.
- Yun:** 28
- Teacher:** So how did you get 28? I think I know what you did but can you share it with everybody?
- Yun:** So seven is like not half but 14 is half so 14...
- Teacher:** Okay. Did you hear what he just said? So if he says that 14 is half. What would seven be?
- Audrey:** 25 percent.
- Teacher:** 25 percent. I love that. So then I'm wondering how he got 28?
- Audrey:** (shows 4 fingers) Four times.
- Teacher:** Say it.
- Audrey:** So four times seven equals 28.
- James:** It's like quarters.
- Teacher:** Do you guys agree with him though? You think 14 would be about half.
- Audrey:** Four times seven equals...
- James:** 'Cause 14 equals 28.
- Teacher:** 14 times 2 equals 28. And do you think 14 is half? Decide if that's what you guys are going to go with. I like where your heads are at, guys, I like how you're being a little bit flexible. I'm going to go check.

Students engaged in community-based mathematics modeling and group work through several key practices. In this excerpt, we coded many instances of how the students were working together to make sense of the mathematics and making assumptions based on key information they had about the small can of paper rolls. First, they participate in *collaborative thinking around identifying important quantities and making assumptions*, using visual cues and partial data to make sense of the problem. For example, Selena observes that the can is "about less than half" full, and Yun builds on this by reasoning, "seven is like not half but 14 is half so 14," demonstrating proportional thinking and idea-sharing. This leads students *engaged in negotiation and consensus building*, as prompted by the teacher's directive: "You guys need to come up with one answer. Okay. Well, let's talk as a group instead of just..." which *encourages collective decision-making*. The students continue on with Yun's idea and start to listen intently, and connect how 7 is 25% or quarter and that 7 times 4 is 28 leading to the group understanding Yun's solution. *Peer-to-peer learning* is evident when students support and refine each other's reasoning, such as Audrey prompting, "Do the math again," and using her fingers to model multiplication. Lastly, students *use informal mathematical language to connect abstract ideas to familiar concepts*, as seen when James relates the problem to "quarters." These interactions highlight how students collaboratively construct mathematical understanding in a supportive group setting.

The teacher uses several facilitative moves to support students in mathematical modeling and group work. One key strategy is *prompting deeper thinking through open-ended questions that encourage students to explain and justify their reasoning*. For example, the teacher asks, “How would you describe how much is being filled right now?” and later, “So how did you get 28? I think I know what you did but can you share it with everybody?” These prompts guide students to articulate their thought processes. The teacher also *encourages participation by inviting quieter students to contribute*, as seen in the prompt, “So Yun I haven’t heard your voice yet.” Additionally, the teacher *validates student thinking by affirming their ideas and promoting flexibility*, saying, “I love what you are saying there, yeah,” and “I like where your heads are at, guys, I like how you’re being a little bit flexible.” Finally, the teacher *scaffolds mathematical concepts by helping students connect informal reasoning to formal mathematical ideas*. This is evident when the teacher asks, “What would seven be?” and builds on Audrey’s response of “25 percent” with affirmation and further discussion. These moves collectively foster a supportive and intellectually rich environment for collaborative mathematical exploration.

Case 2: Creating the Most Inclusive Playground

The *Inclusive Playground* lesson unfolded over four class sessions, engaging third-grade students in the process of designing an accessible and inclusive play space. The instructional sequence was structured to integrate mathematical modeling, social awareness, and collaborative problem-solving. Days 1 and 2 focused on building background knowledge and fostering empathy. Students analyzed data on playground accessibility and shared their preferences for various types of playground equipment. This discussion created a collaborative environment for brainstorming inclusive design ideas. To deepen their understanding, the teacher introduced a “notice and wonder” prompt using images of inclusive play structures. Students were encouraged to consider who might be excluded from traditional playgrounds and what features would ensure access for all. A video on inclusive playgrounds and a poster illustrating the concepts of exclusion, segregation, integration, and inclusion further supported this exploration. One student insightfully noted, “What if it rains? How are people going to ride on the mulch? Because if it rains, the mulch has mud under it, so the wheelchair can get stuck.”

On Day 3, the teacher introduced a real-world constraint: a \$100,000 budget for the playground design. To facilitate student learning, Ms. W. asked the students, “What do you know? Need to know. What are some things that we should consider and can decide on?” This graphic organizer (see Figure 6) became a tool that teachers in our project used to help make sense of the modeling situations and elicited assumptions and variables important to the problem at hand.

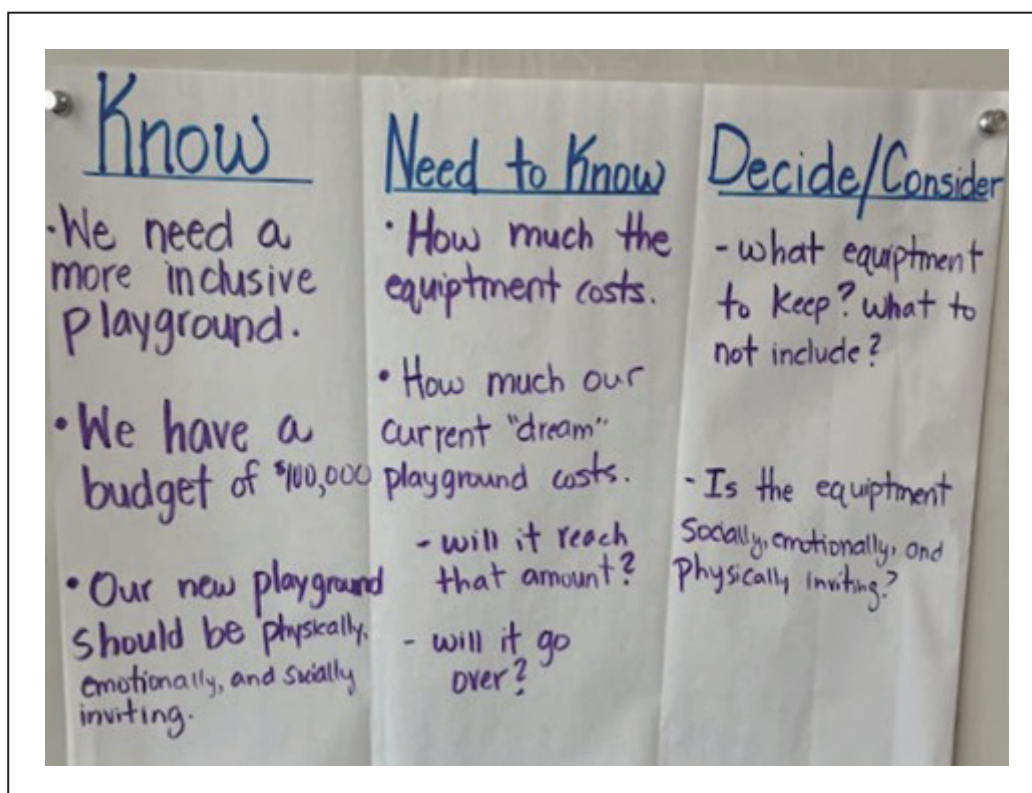


Figure 6. Know, Need to Know, Decide/Consider Graphic Organizer

Excerpts from budgeting for an inclusive playground

- Teacher:** So this is what you're going to do—there's a *bit of a catch, a little curveball*. You'll be working with a partner to design an inclusive playground that fits within a new budget. What does that mean? What challenges might you face?
- Esa:** That you have to subtract.
- Teacher:** Okay, so maybe the math might be a little tricky. What other challenges might come up when working with a partner?
- Don:** The budget is not enough.
- Teacher:** Right. So if you take your dream playground and your partner's dream playground, and you total up all the equipment, you might realize it goes over budget. What would you do then?
- Liz:** Just use one.
- Teacher:** Okay, Liz suggests using just one design. But what if I really like mine, and my partner really likes theirs? What then?
- Isabella:** We might have to take out some stuff.
- Teacher:** So you might need to decide what equipment to keep. I don't want to say "get rid of"—maybe "not include." These could be some tough decisions. If the budget goes over, you'll need to decide what's essential and what can be left out. What else might be challenging?

Student 5: What do we need to consider when we're taking out equipment that's good—like, equipment that you really want?

Teacher: Great question. What should the equipment be that we're keeping? Can we use those three principles again? Is the equipment socially, emotionally, and physically inviting? I'm coming around with your catalogs. Everyone should be seated at a spot. All right, go ahead and get started.

Through group work, students were tasked with designing an inclusive playground within a new budget. They discuss potential challenges, such as the budget being insufficient and the difficulty of deciding which equipment to include or exclude. Ms. W suggests prioritizing equipment that is socially and emotionally inviting. They consider strategies like each partner choosing essential items or compromising on preferences. A catalog with prices was introduced, highlighting the affordability constraints. The team is then paired up to start their design, focusing on what equipment is most important to them.

Introducing a constraint—such as a budget limit—into a mathematical modeling task is a powerful pedagogical move because it mirrors the complexity and unpredictability of real-world problem-solving. In the Inclusive Playground lesson, the teacher introduces this constraint as “a bit of a catch, a little curveball,” prompting students to revise their initial models and make decisions based on limited resources. This shift requires students to engage in deeper reasoning, prioritize essential features, and negotiate trade-offs, all of which are core components of authentic modeling.

For example, when the teacher asks, “What would you do then?” If the combined designs exceed the budget, students respond with strategies like “just use one” or “we might have to take out some stuff.” These responses show students grappling with the need to revise their models in light of new constraints—an essential part of the modeling cycle. As one student insightfully asks, “What do we need to consider when we're taking out equipment that's good—like, equipment that you really want?”—they are not only considering mathematical trade-offs but also the emotional and social implications of their decisions.

This kind of revision process is central to mathematical modeling because it reflects how models are developed and refined in real-world contexts, where variables are interdependent and constraints are common. According to Blum and Leiß (2007), effective modeling tasks should include opportunities for students to iterate and adapt their models, as this fosters a deeper understanding of both the mathematics and the context. Moreover, incorporating constraints like budgets helps students develop modeling competencies that are essential for responsible citizenship and real-life decision-making. By embedding a realistic constraint, the teacher transforms the task from a simple design exercise into a dynamic, authentic modeling experience—one that requires students to balance mathematical reasoning with empathy, collaboration, and critical thinking skills.

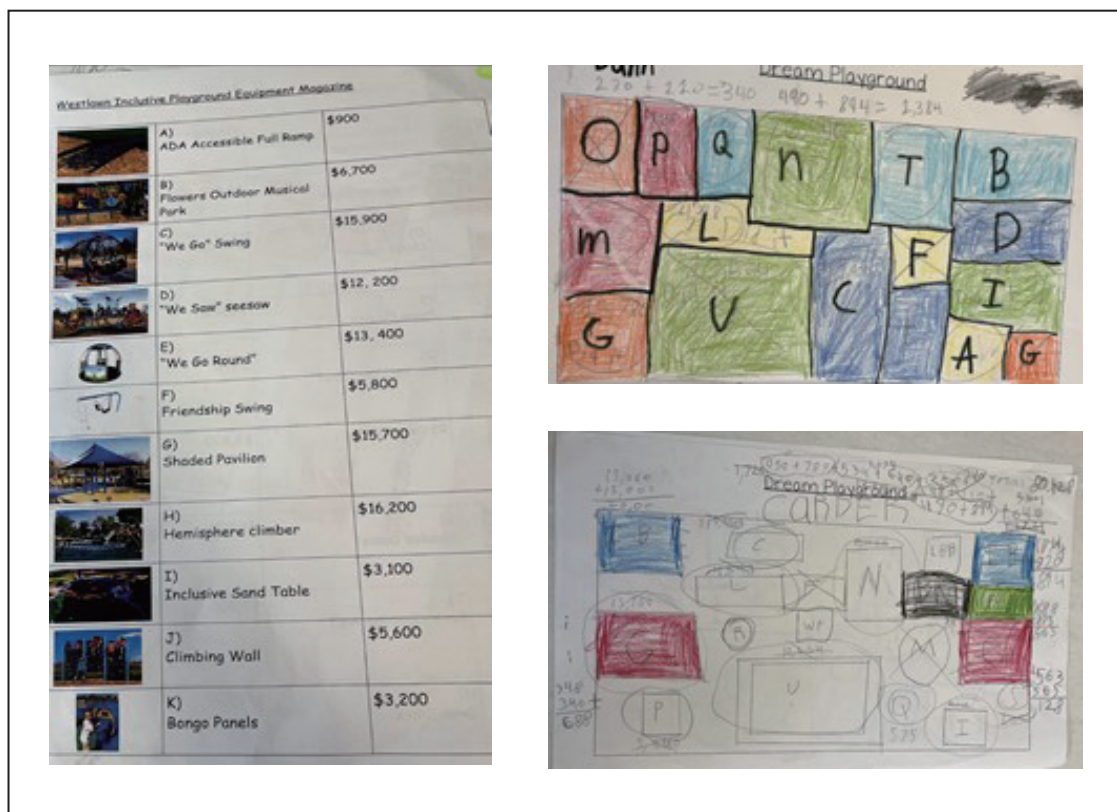


Figure 7. Catalogue and playground blueprint with selections of equipments

Inclusive Analysis Problem and Dialogue

Modeling Task: Suppose you have \$100,000 and wish to build an inclusive playground. What would you include? What factors do you consider when deciding which equipment to include?

Tommy: Perfect. Let's circle the items we're keeping and finalize our list (see Figure 7).

Don: If it's over \$10,000, let's skip it for now. We can get one or two big items, but let's plan carefully. We'll add everything we want first, and if we have money left over, we can go back and add something extra.

Tommy: Do we have item J? I think we need it—it's socially inviting.

Don: Yeah, J is important. It's kind of open and accessible. I'll start calculating the capacity.

Tommy: Let's include two shaded pavilions. How much are they?
\$15,000 for both. One big pavilion should work.

Don: Let's write that down and do the math on paper.

Tommy: Okay, what's really expensive that we can take out? Take out item F—it's not essential.

Don: We definitely need item I. And one shaded pavilion at least.

- Tommy:** What about the tic-tac-toe panel? It's socially inviting, but we don't need two of the same thing.
- Don:** Agreed. Let's just keep one. Now, how much money do we have left?
- Tommy:** The We-Go swing is kind of like a seesaw. It's expensive, though. Maybe we don't need both the swing and the merry-go-round.
- Don:** Let's just keep one swing. Two is too much. And U is \$10,000. That's not bad.
- Tommy:** We already have U. What about the rock climbing wall?
- Don:** It's not very socially inviting. Maybe we skip it for now.
- Tommy:** Safety features are a must. Let's keep those.
- Don:** What about chat noodles? I have two on my list.
- Tommy:** Let's add everything to one list and total it up.
- Don:** Okay, let's double-check. We've got the oval track, the shaded pavilion, the We-Go swing, the sand table, the wheelchair-accessible merry-go-round, the flower musical park, the cozy cocoon, and the squishy floor.
- Tommy:** That's a solid list. How much have we spent?
- Don:** We're at \$96,000. We have \$4,000 left.
- Tommy:** Let's pick one last item for \$4,000. What should it be?
- Don:** Let's go with the squishy floor. It's fun and safe.

Tommy and Don demonstrated thoughtful and sophisticated decision-making as they selected playground equipment based on the criteria of being socially, emotionally, and physically inviting. Their conversation reflects a deep understanding of inclusivity and empathy, which are essential components of effective mathematical modeling in real-world contexts. For example, Tommy emphasized the importance of item J, stating, “Do we have item J? I think we need it—it’s socially inviting,” showing his awareness of how certain features can foster social interaction. Don agreed, noting that J was “kind of open and accessible,” and later added, “We definitely need item I,” reinforcing their shared commitment to inclusivity. Throughout their planning, they evaluated equipment not just by cost but by its potential to serve diverse users. They discussed removing less essential or redundant items—like skipping the Wi-Fi feature and limiting duplicate equipment such as swings—so they could prioritize features that met their criteria. When considering the rock climbing wall, Don remarked, “It’s not very socially inviting. Maybe we skip it for now,” showing how they weighed emotional and social accessibility alongside physical features. Their approach also reflected strategic budgeting. Don suggested, “If it’s over \$10,000, let’s skip it for now,” and they agreed to revisit items if funds remained. This iterative process of adding, removing, and reassessing equipment illustrates how they organized variables and refined their model based on both mathematical constraints and empathetic reasoning. Their final list included items like the wheelchair-accessible merry-go-round, cozy cocoon, and squishy floor—choices that balanced cost with inclusive design. Tommy and Don’s modeling process was grounded in empathy, collaboration, and mathematical reasoning. Their decisions were not only mathematically sound but also socially conscious, reflecting a sophisticated understanding of how to design a playground that is welcoming and accessible to all. During the Inclusive Playground lesson, students engaged in community-based mathematical modeling (CBMM) by collaboratively designing an inclusive playground within a limited budget.

They applied mathematical reasoning to estimate costs, make trade-offs, and prioritize accessible features. Student comments like “you have to subtract” and “the budget is not enough” reflected their awareness of financial constraints. As they worked in pairs, they negotiated design decisions, balancing personal preferences with community needs. The task integrated inclusive design principles and fostered civic-minded problem-solving, as students used math to advocate for equity in their school environment.

The teacher employed several facilitative strategies to support students’ mathematical modeling and collaboration. One key move was to use the graphic organizer “*Know, Need to Know, Decide/Consider*” to support their thinking around important variables like cost of item and space for the equipment. Another key move was introducing a real-world constraint—a reduced budget—which the teacher framed as “a bit of a catch, a little curveball.” This prompted students to think critically about trade-offs and prioritize essential features. The teacher used open-ended questions to guide reflection and encourage deeper thinking, asking, “What challenges might you face?” and later, “What should the equipment be that we’re keeping?” These prompts helped students consider both mathematical and ethical dimensions of their decisions. The teacher also modeled strategies for compromise, suggesting, “We could go with Liz’s idea—say, ‘I’ll give up mine, just use yours.’ Or, we could each pick a few things... and then compare.” By offering multiple approaches, the teacher supported flexible thinking and collaborative negotiation. Additionally, the teacher reinforced the importance of inclusive design by reminding students to use the three guiding principles—social, emotional, and physical accessibility—when making decisions. These facilitative moves created a supportive and intellectually rich environment that empowered students to engage meaningfully with mathematics while addressing a real community need. Students used a photo inventory of playground equipment to assess accessibility and cost. Through group math talk and collaborative decision-making, they tracked expenses, evaluated trade-offs, and prioritized features that would ensure inclusivity. The teacher facilitated this process by prompting students to revise their assumptions and justify their mathematical reasoning. As the teacher noted, “They realized that they had to work together to accomplish this. There was a lot of good conversation about access, and a lot of sense-making.”

Day 4 culminated in a presentation preparation, where students learned that their designs would be shared with the school principal and renovation team. This authentic audience elevated the importance of their work and reinforced the value of student voice in shaping community spaces. As one student reflected, “This is not something that our school can afford right now, but it will help us all view our playground differently. And if that money comes, maybe [the principal] would consider something to help with our inclusive playground.” Throughout the lesson, students engaged in mathematical modeling by applying budgeting, estimation, and spatial reasoning to a meaningful, real-world problem. The integration of group work, teacher facilitation, and authentic constraints created a rich context for developing both mathematical understanding and civic responsibility collaborating on community projects or initiatives.

Table 2. Summary of student engagement in CBMM and teacher facilitation moves

Student Engagement through CBMM	Teacher Facilitation to Support Student Learning
1. Collaborative Math Modeling - Students work in groups to identify important quantities, make assumptions, build and revise their models. - Use of visuals, estimation, computation, and proportional reasoning. 2. Peer-to-Peer Learning - Students build on each other’s ideas (e.g., “group-think” math talk). - Informal language and analogies (e.g., “quarters”) support understanding. 3. Negotiation and Consensus Building - Students discuss and revise ideas to reach group decisions. - Encouraged to justify reasoning and consider multiple perspectives. 4. Real-World Relevance - Tasks rooted in authentic community issues (e.g., pollinator habitats, inclusive playgrounds). - Students connect math to civic responsibility and sustainability.	1. Use of Open-Ended Questions - The teacher prompts deeper thinking. 2. Encouraging Equitable Participation - The teacher invites quieter students to contribute and validate diverse ideas. - Emphasis was placed on equitable participation, ensuring all voices. 3. Scaffolding Mathematical Thinking: - The teacher used sentence stems “<i>If I know __, then I could __</i>” and graphic organizer “<i>Know, Need to Know, Decide/Consider</i>” to guide assumption making and model development. - The teacher helped students connect informal reasoning to math concepts. 4. Introducing Real-World Constraints: - The teacher introduced a budget constraint as a “curveball,” prompting students to revise their models and make trade-offs—mirroring real-world decision-making. 5. Modeling Civic and Empathetic Reasoning: - Teachers encouraged students to consider social, emotional, and physical accessibility reinforcing human-centered decision-making in modeling.

Discussion and Future Direction

The findings from this study highlight the transformative potential of community-based mathematical modeling (CBMM) in fostering both mathematical understanding and civic engagement among elementary students. Students engaged in collaborative, inquiry-driven modeling tasks that emphasized critical thinking, empathy, and social awareness. In the Bee Hotel task, students like Yun, Audrey, and James demonstrated proportional reasoning and estimation as they worked together to determine how many paper rolls would fill a can. Their group dialogue reflected authentic mathematical modeling, marked by conjecturing, critiquing, and revising ideas through peer-supported reasoning. Similarly, in the Inclusive Playground task, students such as Tommy and Don made thoughtful decisions that balanced budget constraints with inclusive design principles. Their choices were informed not only by mathematical efficiency but also by a deep consideration of emotional, physical, and social accessibility.

These modeling experiences illustrate how empathy can guide variable selection, constraint negotiation, and final decision-making, enhancing both the mathematical and human relevance of students’ models. Drawing from the literature on culturally responsive pedagogy and mathematical modeling (Suh et al., 2023; Kaiser & Brand, 2015), this study shows that embedding real-world, socially relevant problems into the mathematics curriculum can deepen students’ collaborative problem-solving skills and civic awareness. Ultimately, these practices cultivate civic reasoning and discourse, as defined by Lee et al., (2021), the ability to engage in respectful dialogue, weigh evidence, and make informed decisions for the common good. Through CBMM, students are not only learning mathematics—they are learning how to be thoughtful, empathetic, and civically engaged members of their communities.

The teacher played a pivotal role in fostering student engagement and supporting equitable participation through strategic facilitation that promoted critical thinking, empathy, and meaningful discourse. In both case studies, they used open-ended questions and structured sentence stems such as “If I know __, then I could __” to scaffold students’ reasoning, helping them articulate assumptions, connect informal ideas to formal mathematics, and revise their models based on peer feedback and real-world constraints. The teacher also encouraged diverse perspectives and invited quieter students to contribute, cultivating inclusive classroom environments. A particularly impactful move was the introduction of a budget constraint in the Inclusive Playground task, which prompted authentic decision-making and iterative model refinement. These facilitative strategies align with culturally responsive teaching practices outlined by Zavala and Aguirre (2024), demonstrating how intentional instructional design can create intellectually rich spaces where students explore mathematics as a tool for civic engagement and social transformation.

This study contributes to the growing body of research that positions mathematical modeling as a powerful approach for integrating academic rigor with social relevance. Through group discourse, negotiation, and iterative model refinement, students develop the ability to reason critically, communicate effectively, and make informed decisions. These experiences mirror the demands of real-world problem-solving and align with global educational goals, such as the UN Sustainable Development Goals and Education for Sustainable Development (ESD).

Facilitating mathematical modeling in the classroom presents several challenges, including managing open-ended tasks, supporting diverse student thinking, and guiding collaborative discourse without over-directing the process. Many teachers may feel underprepared to navigate the ambiguity and complexity that modeling tasks often entail—particularly when students are expected to make assumptions, negotiate variables, and revise their models in real time. However, the teacher in this study exemplified key facilitative practices that made modeling accessible and meaningful for all students. Through the use of open-ended questions, sentence stems like “If I know , *then I could* ,” and intentional prompts that encouraged negotiation and consensus-building, the teacher created a space where students could engage deeply with both the mathematics and the social context of the task.

In terms of related work and future directions, I am currently exploring, in partnership with researchers Yao and Zhang, the promise of generative AI in collaborative problem solving and modeling (Yue et al., 2025). As collaborative problem-solving is recognized as a critical 21st-century skill (OECD, 2017), our research investigates how students can co-construct mathematical models alongside AI agents designed to scaffold reasoning, support consensus-building, and enhance group decision-making. Generative AI has the potential to act as a peer collaborator—posing reflective questions, offering alternative representations, and prompting students to consider diverse perspectives. For example, AI can help students identify important variables, simulate outcomes based on different assumptions, and visualize the impact of their decisions in real time. This not only strengthens students’ modeling competencies—such as assumption-making, model refinement, and interpretation—but also nurtures empathetic and civic reasoning by encouraging students to weigh the social and ethical implications of their models. In addition to supporting students, generative AI can also serve as a powerful tool for teacher learning and practice. Facilitating mathematical modeling requires skillful questioning, scaffolding, and responsiveness to student thinking—practices that can be challenging to develop. AI-powered coaching tools could provide teachers with real-time suggestions for discourse moves, model effective facilitation strategies, or simulate classroom scenarios for professional learning. By helping teachers rehearse and refine their facilitative practices, AI can support the broader implementation of CBMM and ensure that more classrooms become

spaces where students engage in rigorous, socially meaningful mathematics. Through AI-enhanced collaboration and facilitation, educators can cultivate a generation of learners who are not only mathematically proficient but also empowered to use mathematics as a tool for social change, innovation, and collective problem-solving.

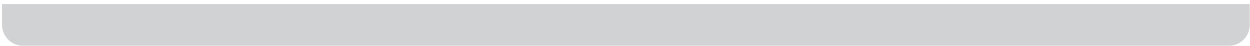
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TEACHING MATHEMATICS THROUGH PROBLEM POSING: A SYSTEMATIC REVIEW OF PROBLEM-POSING-BASED LEARNING RESEARCH

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Problem posing (PP) has been widely recognized in mathematics education as a student-centered instructional approach that enhances cognitive and affective learning aspects. Research has shown that incorporating PP into mathematics instruction deepens students' conceptual understanding and mathematical creativity while fostering their engagement and sense of ownership in learning. As various curriculum documents emphasize, PP involves formulating and reformulating mathematically rich problems, providing learners with opportunities to construct and apply knowledge actively. This lecture will present findings from a recent systematic literature review focusing on PP-based learning (P-PBL) in P-12 mathematics classrooms. The review documents what P-PBL looks like in practice and how its various characteristics have been studied within mathematics education research. Drawing on a best-evidence approach, we will conclude by offering recommendations for researchers, teachers, and teacher educators interested in advancing P-PBL practice and research.

Keywords: problem posing, problem-posing-based learning, teaching mathematics through problem posing, P-PBL, mathematical problem posing

Introduction

Problem posing (PP) is a key mathematical practice that involves formulating or reformulating new mathematical problems based on given situations (Cai & Hwang, 2020; Silver, 1994). PP has been conceptualized as an integral part of problem solving (Pólya, 1954), requiring problem posers to generate new problems by making variations to existing ones, such as altering the given conditions, context, or outcomes (Brown & Walter, 1983). PP can serve two primary roles in classroom instruction: as a *tool* and a *goal* of mathematics learning (Cai & Leikin, 2020; Kilpatrick, 1987). As a *tool*, teachers can use PP to deepen students' content knowledge (e.g., Koichu, 2020) and foster mathematical creativity (e.g., Matsko & Thomas, 2015). As a *goal*, PP activities aim to develop students' competency in posing meaningful mathematical problems (e.g., Leikin, 2015). As there is a growing body of research on the roles and benefits of PP in mathematics education, some research has recently explored the nature and effects of PP processes and interventions (Cai & Hwang, 2020). However, despite the increasing interest in PP-based learning over the past 15 years (Cansiz Aktas, 2022), relatively little work has synthesized how mathematics can be taught through problem posing, referred to as PP-based learning (P-PBL).

Previous research has primarily focused on the effects of PP interventions on cognitive and/or affective outcomes. However, there remains a lack of systematic understanding of P-PBL as an instructional approach for students and teachers. Although a few review studies have emerged, such as systematic literature reviews (e.g., Lee, 2021; Zhang et al., 2025) and meta-analyses (e.g., Ran

et al., in press; Wang et al., 2022; Zhang et al., 2024), there is still a significant gap in understanding how P-PBL is designed, what it looks like in practice, and what impacts it has on teaching and learning mathematics. Given the increased emphasis on providing students with rich opportunities to engage in PP (Cai et al., 2015), a systematic review of empirical studies on P-PBL is timely and necessary. Such review work, documenting various components and designs of P-PBL and identifying key findings across studies, will highlight best-evidence recommendations to inform future research and practice (Moher et al., 2010). This review paper addresses the following research questions:

1. What does P-PBL look like in P–12 mathematics classrooms?
2. What characteristics of P-PBL have been studied and reported in the mathematics education literature?
3. What recommendations and approaches can support more effective and sustainable implementation of P-PBL for teachers?

Teaching Mathematics through Problem Posing

Teaching mathematics through PP, or P-PBL, refers to instruction integrating PP tasks into the learning process. In mathematics education research, P-PBL has supported various pedagogical goals: deepening students' mathematical understanding, fostering sensemaking, promoting creativity, connecting mathematics to students' lived experiences, or developing students' mathematical identities (Kilpatrick, 1987; Silver, 1994). Effective implementation of P-PBL requires careful planning and instructional preparation from teachers (Cai et al., 2024). Because teachers often serve as the initial problem posers, their decisions on task design directly shape students' learning opportunities (Butts, 1980; Li et al., 2020), which greatly relies on teachers' knowledge and beliefs. According to Cai et al. (2024), teacher learning involves the development of knowledge and beliefs, which inform practice and ultimately influence student outcomes. Despite the benefits of students' engagement with PP tasks, many teachers still find P-PBL relatively unfamiliar and often struggle with how to plan and implement practical PP tasks.

Meta-Research on P-PBL Literature

To better understand what works in P-PBL, recent meta-research, including systematic reviews and meta-analyses, has focused on the effectiveness of PP interventions. For example, Wang et al. (2022) synthesized studies with quantitative measures such as effect sizes for intervention components and moderators. Zhang et al. (2024) also conducted a meta-analysis of 26 PP intervention studies, identifying nine common instructional components: Overview of PP (WPP), Discussion of “good” problems (WGP), Engagement in PP activities (PPA), Evaluation of posed problems (EPP), Comprehension of problem situations (CPPS), Strategy use (SPP), Examples (PPE), Technology use (TPP), and Interactive environments (ILE). They found a medium, statistically significant effect of PP interventions on PP competence, with more substantial impacts on noncognitive outcomes than on the quality or quantity of posed problems. In a follow-up review, Zhang et al. (2025) analyzed 39 studies aiming to develop students' PP competence. The learning outcomes were categorized as: a) quantity of posed problems (e.g., correctness, appropriateness), b) quality of posed problems (e.g., complexity, difficulty), and c) affective-motivational factors (e.g., beliefs, attitudes, familiarity). However, these reviews focused narrowly on the studies with PP interventions targeting PP competence and PP-related competences (e.g., problem-solving competence). The broader landscape of P-PBL—including studies focusing on problem solving, creativity, or classroom-level instructional

models—remains underexplored. Our work builds on these reviews by synthesizing a wider range of P-PBL literature to understand better how P-PBL has been used and studied across mathematics classrooms.

P-PBL Instructional Model

Though much of the research on PP has focused on its cognitive and affective impacts on students (Cai & Leikin, 2020), several studies have also aimed to conceptualize the processes involved in students’ PP activities. For example, Cai and Rott (2024) proposed a general model outlining four phases of the PP process: orientation, connection, generation, and reflection. Similarly, Baumanns and Rott (2022b) described an iterative model comprising five phases—situation analysis, generation, variation, problem solving, and evaluation—highlighting the dynamic nature of student engagement with structured PP situations.

Although these models capture the cognitive processes involved in posing problems, relatively few researchers have examined what teaching mathematics through PP entails in classroom practice. To address this gap, Cai (2022) introduced the P-PBL instructional model (Figure 1), which outlines four instructional steps. The fourth step is especially critical: How teachers use and respond to students’ posed problems can significantly shape the task’s learning experience and instructional value. In the following sections, we draw on findings from our ongoing scoping review to examine these four steps, focusing on implementation variations and their implications for supporting student learning through PP.

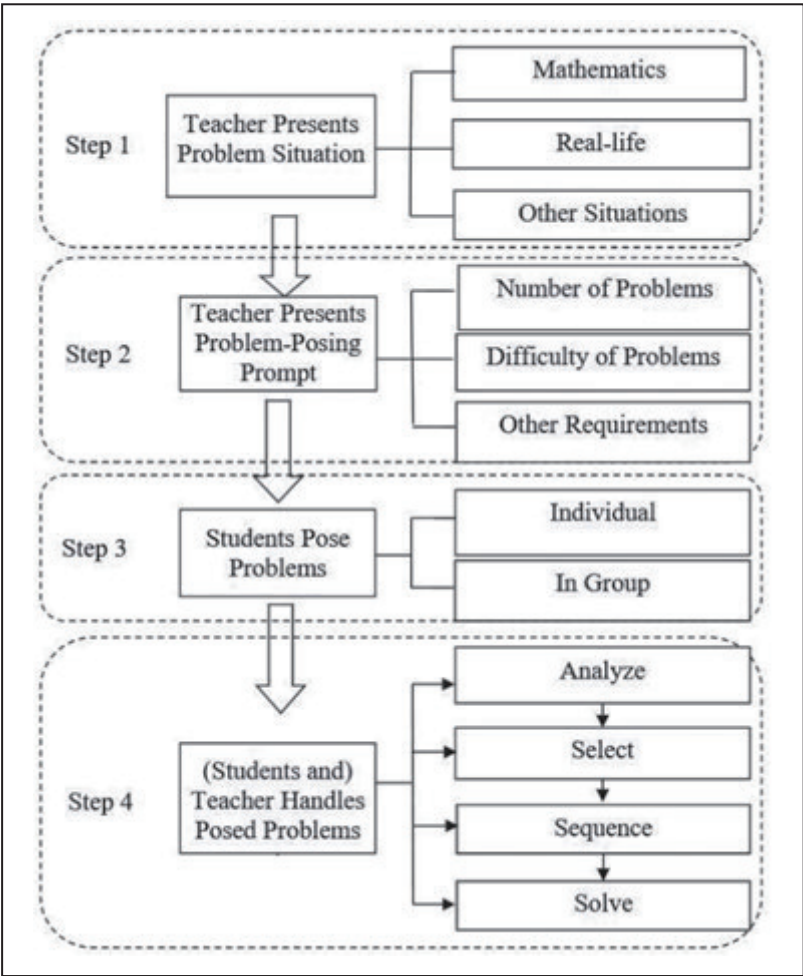


Figure 1. P-PBL instructional model (Cai, 2022)

Problem-Posing Tasks

Stoyanova and Ellerton (1996) categorized PP tasks into three types based on structure. The most constrained PP task type is *structured*, which asks posers to pose problems by modifying an existing problem (e.g., “Change the numbers in this problem to create a new one”). The second most constrained PP task type is *semi-structured*, which provides a situation but no model problem, requiring students to generate questions from the context. Last, the least constrained PP task type is *free*, offering no restrictions, allowing students to pose problems freely based on what they have learned or interests them.

Although the terms *PP situation* and *PP task* are often used interchangeably in the literature, we distinguish them as follows: PP tasks consist of both (a) a *problem situation*—a (non)mathematical context that serves as the foundation for problem generation—and (b) a *PP prompt*—clear instructions that guide what the poser is expected to do (Cai & Hwang, 2020). In this paper, we use the term *problem situation* instead of *PP situation* to refer specifically to a (non)mathematical context that provides information for students to generate problems. Unlike *PP situations*, which sometimes imply context and action (as defined in Stoyanova & Ellerton, 1996), *problem situations* in our framework provide only the background or stimulus for posing (Cai & Hwang, 2020). These can range from mathematical objects like equations or scatter plots to real-world contexts like grocery stores, farms, or even common artifacts like brochures or newspapers.

Prior studies suggest that the type of problem situation can influence students’ ability to pose meaningful problems. Whereas some contexts support creativity and engagement, others, particularly real-life scenarios lacking numerical data, can present challenges for problem generation. In contrast, *PP prompts* specify what students are expected to do with a given situation. These prompts may include how many problems to pose, constraints to follow, or levels of difficulty to aim for (e.g., “Pose a mathematical problem that would challenge your classmates”). Compared to the problem-solving literature, relatively little research has examined how different elements of PP tasks influence students’ PP competence.

Baumanns and Rott (2022a) proposed a framework for analyzing PP tasks using three dimensions: (a) generating new problems versus reformulating existing ones, (b) posing routine problems versus nonroutine problems, and (c) the degree of metacognitive engagement involved. Although they did not explicitly distinguish PP prompts from PP situations, their framework offers a valuable perspective for examining variation in PP task design. In another review study, Baumanns and Rott (2022b) highlighted the importance of understanding a given problem situation—identifying and modifying conditions within a given problem—as a fundamental part of creating new problems. Drawing on these insights, Cai and Rott (2024) emphasized that understanding the structure of a PP task, particularly the problem situation and associated prompts, is a critical first step in the PP process and lays the groundwork for effective implementation of P-PBL in mathematics classrooms. However, recent research has begun to investigate the influence of the structure of PP prompts more systematically. For example, Cai et al. (2023) investigated how PP prompt variations impact students’ performance. They used prompts with varying levels of specificity, such as:

- Pose three mathematical problems that can be solved using the information in the given problem situation
- Pose one easy, one moderate, and one complex problem
- Pose one problem similar to and one very different from a sample problem
- Pose a mathematical problem that can challenge your classmates/teacher

Their findings suggest that students generate more linguistically and semantically sophisticated problems when responding to more structured and targeted prompts.

Advances in P-PBL Research in Mathematics Education

The reviewed P-PBL studies identified significant variations in how teachers and researchers designed and implemented student PP experiences. These variations can be understood along four key dimensions: problem situations, PP prompts, settings for student PP, and handling of posed problems. Real-life contexts were the most used problem situations, ranging from everyday word problems to authentic artifacts such as brochures and menus. Studies employed a wide range of prompts—from open-ended invitations to pose any problem of interest to structured prompts requiring students to apply specific strategies or reformulate given problems. Most studies asked students to pose problems individually, but several implemented more unique formats, including whole-class collaborative posing, one-on-one student-teacher interactions, or game-based systems. Finally, although only a few studies explicitly focused on how posed problems were used, some documented meaningful follow-up activities such as peer solving, feedback exchange, cross-classroom collaboration, and revision processes. The following sections explore these four aspects in greater depth, highlighting patterns, pedagogical intentions, and implications for future P-PBL design.

Problem Situations in P-PBL Studies

Figure 2 presents the distribution of problem situations used across the P-PBL studies we reviewed. The most common type was real-life situations, which were used exclusively in 15 studies. Eight additional studies incorporated real-life and purely mathematical situations, whereas only one focused solely on mathematical situations. Five other studies used free situations, where students were invited to pose problems based on their interests or prior knowledge, or did not provide sufficient detail about the problem situations used. Notably, two studies (Chang et al., 2012; Suarsana et al., 2019) did not specify the nature of the problem situations. As noted, real-life contexts were the dominant type, appearing in 15 studies exclusively and 23 studies total, including those that used real-life and mathematical elements, accounting for roughly 75% of the studies reviewed. These real-life contexts varied in complexity and form from standard word problems referencing everyday scenarios (e.g., Arikan & Unal, 2014; Kapur, 2015) to the use of authentic artifacts such as grocery store flyers or theme park brochures (e.g., Bonotto, 2013).

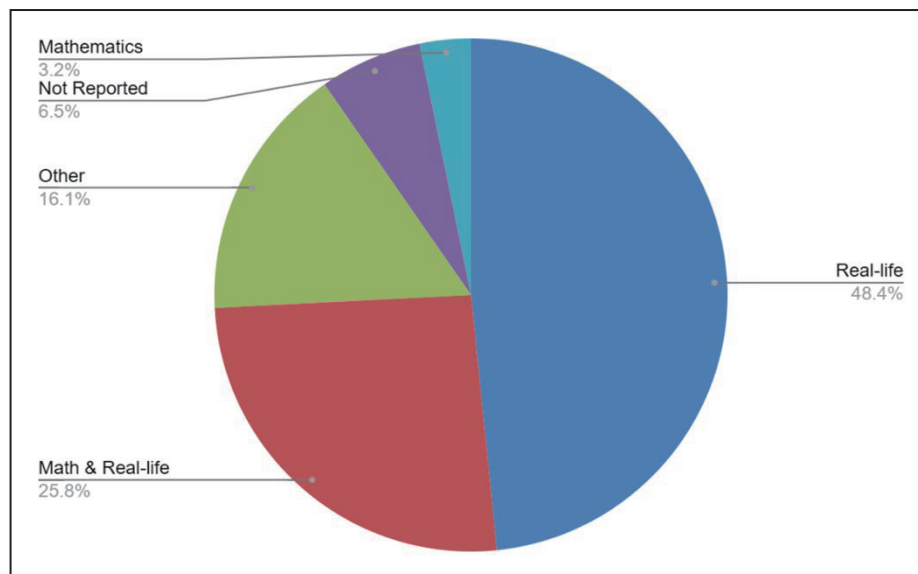


Figure 2. Types of problem situations in P-PBL studies

The second most common approach was blending real-life and mathematical situations, as seen in eight studies. In contrast, only one study (Cheng et al., 2014) focused solely on mathematical situations, which ranged from formal mathematical expressions (e.g., $8 \times 12 = ?$) to annotated equations. In the following sections, we unpack selected studies to illustrate how different types of problem situations were integrated into P-PBL instruction and their roles in students' learning experiences.

Real-Life Situations

Real-life situations utilized in the P-PBL studies also had some variations. While we read through the P-PBL studies, we were able to further categorize the types of real-life situations utilized in the studies, including but not limited to real-life context-infused situations, such as pictures, tables, or graphs (e.g., Bicer et al., 2020; Kapur, 2015, 2018), real-life artifacts (e.g., Bonotto, 2013), or mathematical stories and literature (e.g., English, 1998). The most common type of real-life situation was word problems, including to some extent real-life contexts. For example, Bicer et al. (2020) examined P-PBL's effects on elementary students' mathematical creativity and self-efficacy using real-life contexts. Over 4 months, students participated in 20-minute semi-structured PP activities twice a week. The problem situations included real-life elements such as pictures, tables, graphs, number sentences, and equations. Each session began with students describing what they observed in the given problem to build familiarity. They then individually posed problems based on these contexts. Preservice teachers (PSTs) monitored their progress, posed guiding questions, and supported revisions when necessary. Students were encouraged to evaluate the viability and appropriateness of their posed problems; describe solution strategies; and, if time allowed, generate additional problems and reflect on their similarities and differences. Each session typically covered two PP tasks, ending with a discussion on which problem was the most original to foster a sense of creativity and evaluative thinking. The study found that these P-PBL sessions positively influenced students' creativity in mathematics and that creativity was strongly correlated with their self-efficacy.

Bonotto (2013) conducted two studies exploring the use of real-life artifacts—specifically an amusement park brochure—as problem situations in P-PBL to support students’ PP and problem solving. In Study 1, fifth-grade students participated in two 2-hour P-PBL sessions to investigate how real-life materials influenced their PP and problem-solving abilities and the relationship between these abilities. Students engaged in five key activities: (1) collectively analyzing the brochure and its numerical content; (2) selecting relevant data; (3) creating a problem using that data, including a percentage calculation and two questions; (4) solving another group’s problem; and (5) engaging in a whole-class discussion about possible errors or inconsistencies in the posed problems. All student pairs successfully translated real-life data into mathematics and solved the problems created. In Study 2, students received an amusement park brochure containing information such as group rates, menu options, discounts, and access fees (Figure 3). As a warm-up, they wrote down observations before posing problems. They were then asked to pose as many solvable problems as possible, self-assess their problems, and submit them for peer review. Four student-created problems were selected for further problem-solving activities: one multi-step problem involving discount calculations and group totals, two open-ended pricing problems, and one problem with incorrect data due to misinterpreting brochure conditions (e.g., overlooking a free-entry-for-every-ten rule). The students were first asked to solve the selected problems by writing out their solution strategies and noting the considerations they made during the problem-solving process. At the end of the session, a whole-class discussion encouraged students to explore various solutions and reflect collectively on their problem-solving experiences with the selected problems. It was found that there were differences in students’ mathematical creativity. In particular, the class with a higher proportion of high-achieving students tended to pose more solvable and original problems. These results suggest a need for further research to investigate potential relationships between students’ mathematical performance and their mathematical creativity in PP.

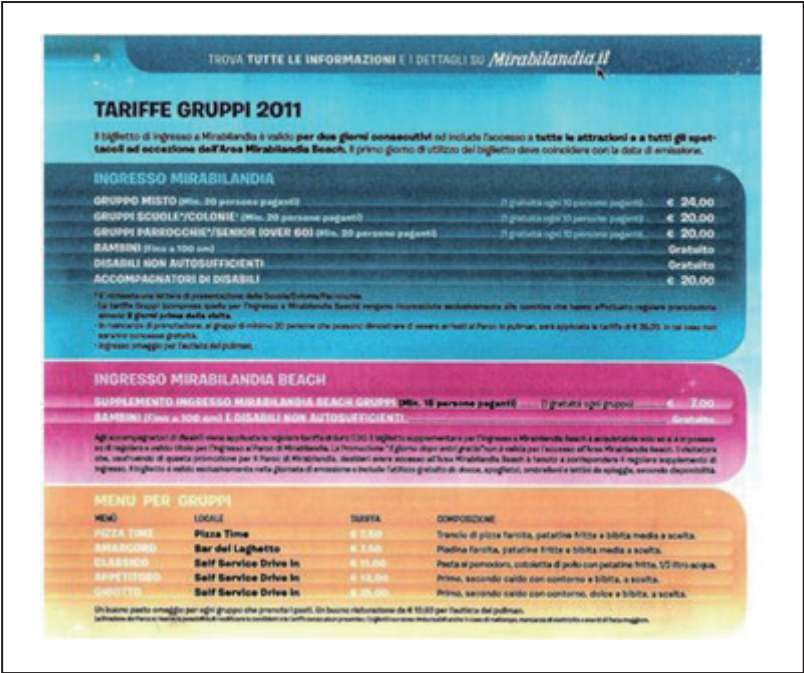


Figure 3. Example of a real-life artifact (Bonotto, 2013)

Mathematical Situations

Although most P-PBL studies incorporated real-life contexts as problem situations, some focused on using mathematical expressions to support the development of students' PP competencies and related skills. For example, Cheng et al. (2014) investigated the effects of using pure and annotated mathematical expressions (Figure 4) on fourth-grade students' problem-solving abilities, specifically their translation, integration, and planning skills.

Pure Mathematical Expression	Annotated Mathematical Expression										
$8 \times 12 = ?$ (a)	<table><tr><td>Number of People per Group</td></tr><tr><td>8</td></tr></table>	Number of People per Group	8	$\times$	<table><tr><td>Number of Groups</td></tr><tr><td>12</td></tr></table>	Number of Groups	12	$=$	<table><tr><td>Total Number of People</td></tr><tr><td>?</td></tr></table>	Total Number of People	?
Number of People per Group											
8											
Number of Groups											
12											
Total Number of People											
?											

Figure 4. Examples of mathematical situations (Cheng et al., 2014)

Over five 40-minute sessions across 3 weeks, students were asked to pose word problems involving the four basic operations. This included four one-step problems for each operation and two two-step problems for each of the following combinations: multiplication–addition, multiplication–subtraction, addition–multiplication, and subtraction–multiplication. One group worked with pure mathematical expressions (Figure 4a), whereas the other used annotated expressions that included contextual cues (Figure 4b). Both groups showed improvement in overall problem-solving scores. However, only the annotated-expression group demonstrated significant gains in the ability to integrate and transform information into word problems as well as in formulating two-step problems. In contrast to Cheng et al.'s focused comparison, most other studies reviewed did not isolate the impact of different problem situations. Instead, they often incorporated real-life and mathematical contexts to enrich the P-PBL experience. These studies generally used real-life situations to position PP as meaningful, applied practice—connecting mathematics to students' lived experiences—rather than as an abstract or isolated classroom exercise.

PP Prompts in P-PBL Studies

Across the P-PBL studies in our review, a wide range of PP prompts were used—some focused on generating new problems, others on reformulating existing ones. Two major dimensions characterize these prompts: (a) the level of structure (ranging from open to highly structured) and (b) the type of action required (generating vs. reformulating). Prompts for generating problems ranged from free form (e.g., "Write a math problem you would like to solve," Lowrie, 2002) to semi-structured (e.g., "Pose a problem such that its solution matches the one above," Arikan & Unal, 2014). Reformulating prompts also varied from open ended (e.g., "Pose a similar problem," Ozdemir & Sahal, 2018) to highly structured instructions (e.g., "Pose a new problem by changing the value but keeping the data," Suarsana et al., 2019). Table 1 presents the types of PP prompts and illustrates examples used across the reviewed P-PBL studies.

Table 1. List of PP Prompts in P-PBL Studies

PP Prompt Type	Examples & Citations
Generating new problems - Free	Students freely generate problems from personal interest or experience without constraints: - Write (or tell me about) a math problem you want to solve. (e.g., Lowrie, 2002) - Create three sets of problems for a school website. (e.g., Manuel & Freiman, 2017) - Pose problems using a system with a given range. (e.g., Chang et al., 2012) - Pose mathematical stories from life related to a topic. (e.g., Peng et al., 2020; Winograd, 1993) - Add words collaboratively to create a problem. (e.g., Cankoy, 2014)
Generating new problems - Semi-structured	Students generate new problems from given contexts or representations: - Use annotated expressions. (e.g., Cheng et al., 2014) - Create problems from figures, tables, stories, or real-life data. (e.g., Arikan & Unal, 2014; Bicer et al., 2020; Ozdemir & Sahal, 2018) - Generate questions from tables, pictures, and literature. (e.g., English, 1998; Kapur, 2015, 2018; Kopparla & Capraro, 2018) - Use knowns, constraints, or numerical answers. (e.g., Chen et al., 2015; English, 1997) - Compose complex stories from simpler ones. (e.g., Rudnitsky et al., 1995)
Reformulating given problems - Semi-structured	Students reformulate a previously solved or provided problem: - Pose similar problems based on earlier tasks or personal interest. (e.g., Ozdemir & Sahal, 2018; Palmér & van Bommel, 2020; Walkington, 2017; Walkington & Bernacki, 2015)
Reformulating given problems - Structured	Students systematically transform a given problem: - Change values, context, or constraints. (e.g., Apari et al., 2022; Suarsana et al., 2019) - Reverse knowns/unknowns, apply Brown and Walter's (1983) "what-if-not" strategy. (e.g., Chen et al., 2015; English, 1997)

We identified 10 P-PBL studies that primarily focused on the effects of specific characteristics of PP prompts. Below, we provide brief summaries of selected studies to illustrate how each implemented and enacted P-PBL instruction within its particular context.

Generating New Problems

One of the most used PP prompts in P-PBL studies involved asking students to generate new problems. Even within this category, the structure of the prompts varied considerably, ranging from entirely open-ended to more guided formats. In some studies, students were free to pose problems of their choosing. For instance, in Lowrie's (2002) study, students were simply asked to "write a mathematics problem they would like to solve" (p. 90). With no limitations on format, topic, or difficulty, students had the opportunity to exercise full agency over the content and structure of the problems they created. This approach empowered students and gave teachers valuable insights into their interests and mathematical thinking.

Other studies offered slightly more direction while still allowing for creativity. In Manuel and Freiman's (2017) study, students were asked to design math problems for a peer-accessible website, introducing a real-world purpose to their task. Similarly, in studies by Chang et al. (2012), Peng et al. (2020), and Winograd (1993), students created story problems based on their own life experiences but within the boundaries of a mathematical topic selected by the teacher. These prompts struck a balance between personal relevance and curricular alignment.

Some studies employed semi-structured prompts in which students were required to base their problems on specific mathematical representations or contexts. For example, students were asked to create a certain number of problems using given figures and tables (Arikan & Unal, 2014), generate as many questions as possible from a data set (Kapur, 2015), or write challenging problems inspired by a story context (Rudnitsky et al., 1995). These prompts provided a scaffolded space for creativity while still grounding students in mathematical thinking.

For example, Rudnitsky et al. (1995) examined how teaching third- and fourth-grade students the structure of story problems through P-PBL affected their problem-solving performance. Across 16 sessions, students worked in pairs on various activities: writing problems for given stories, composing their own mathematical stories with additional details, generating all possible problems from their narratives, and combining multiple story contexts into complex problems. The results showed that students in the structure-plus-writing condition, who had opportunities to create, adapt, and extend problems, significantly outperformed peers who did not engage in these creative PP activities. These studies demonstrate the diversity and instructional potential of PP prompts in P-PBL. These prompts engage students in rich mathematical thinking and help bridge content knowledge with creativity and personal experience.

Reformulating Given Problem(s)

Reviewing the P-PBL studies, we found that a set of PP prompts used in P-PBL studies involved asking students to reformulate given problems. Due to their inherent structure, these prompts were typically semi-structured or structured. Four studies in our review used semi-structured prompts, all of which were nearly identical—students were asked to pose a problem similar to the one provided. For example, Ozdemir and Sahal (2018) investigated the effects of P-PBL focused on integers on students' problem-solving competence and attitudes toward mathematics. A total of 69 sixth-grade students participated in the study. Students in the P-PBL group were asked to create problems similar to those in the lesson plan. Additionally, they were asked to develop problems based on a story or real-life data and then solve the problems posed. Students were also given opportunities to revise their problems based on mathematical validity, potential errors, or level of challenge. The results showed that students in the P-PBL group outperformed their peers on a mathematics achievement test focused on integers. However, the two groups had no significant difference in attitudes toward mathematics. The authors also observed that students in the P-PBL group demonstrated a high level of engagement in the PP process and a noticeable sense of competitiveness when creating problems for their classmates.

Palmér and van Bommel (2020) explored how sixth-grade students interpreted the notion of “similar” when asked to pose problems for their classmates. Students were instructed to create a mathematical problem similar to one they had previously explored. The study included two lessons: one focused on problems solving and the other on PP. In the problem-solving lesson, the class investigated a three-dimensional block structure, working to determine how many blocks were needed to construct it based on a two-dimensional image. As students worked through the task, a class discussion emerged around the concept of “hidden blocks”—those not visible from certain perspectives,

such as a bird's-eye view. In the subsequent PP lesson, students were asked to pose a problem similar to the one from the problem-solving lesson. They were given the option to work individually, in pairs, or in small groups. Each posed problem was analyzed along three dimensions: difficulty level (e.g., inclusion of hidden blocks), task structure (whether the problem mirrored or diverged from the original), and mathematical validity. The results showed that most of the students' posed problems closely resembled the original task in design. Specifically, 12 out of 15 problems included the concept of hidden blocks, and six focused on determining how many blocks were needed to build the structure. The findings suggest that students primarily drew on the three-dimensional aspects of the original problem when constructing their own.

In contrast to studies using open or semi-structured prompts, two P-PBL studies employed structured PP prompts, requiring students to generate new problems by applying specific strategies or techniques. These structured prompts guided students to intentionally manipulate aspects of a given problem. For example, by changing or adding information, altering numerical values, reversing knowns and unknowns, modifying or removing constraints, or changing the problem's context while keeping its values intact. One such study, conducted by Suarsana et al. (2019), explored how different modalities of P-PBL—online, face to face, and non-P-PBL—influenced students' problem-solving performance. In both the online and face-to-face P-PBL groups, students first worked collaboratively to solve a given problem. Then, using a random assignment method, each group was instructed to pose a new problem by applying one of four specific strategies: (a) changing the original information, (b) adding new information, (c) changing the values while preserving the structure, or (d) altering the context but keeping the same values. Once they were assigned ways to pose problems, they created problems and solved the problems of their own creation and those of other groups. Only the students in the online P-PBL class were asked to upload their assignments, which involved individual PP after problem-solving. The teacher provided immediate feedback on the problems they posed. Students then revised their problems based on the feedback. They were also free to continue discussing their assignments or related materials with classmates through the online learning platform. It was found that the students in the online P-PBL class outperformed in problem-solving competence compared to the ones in the P-PBL class or the non-P-PBL class.

Apari et al. (2022) also employed structured prompts to examine the impact of P-PBL integrating dynamic geometry software before and during PP on students' PP competence and their views of PP. Over 13 weeks, 16 eighth-grade students participated in a P-PBL program focused on triangle inequality, guided by a six-stage model: modeling, drawing, locating, problem solving, PP, and discussion. Students completed seven PP tasks using GeoGebra. Each task began with a brief instructional session and demonstration of how to use the software to construct triangles. After hands-on exploration and solving a teacher-provided problem, students were prompted to pose a new problem within GeoGebra by modifying the original task. The structured PP prompt was:

Pose a new problem in GeoGebra software by changing the state, conditions, and assumptions of the triangle inequality problem you have solved above and solve the problem you have posed. (Aparai et al., 2022, p. 102)

Following this, students evaluated each other's problems, discussed the importance of specifying integer side lengths, and compared solution strategies. The findings showed improvement in all dimensions of PP competence, with the most significant gains in solvability and the smallest in originality. Overall, students viewed P-PBL positively, reporting that it enhanced their mathematical thinking, PP, and problem-solving skills despite some challenges with posing or solving their problems.

Interestingly, several P-PBL studies employed multiple types of PP prompts within the same study (e.g., Chen et al., 2015; English, 1997; Ozdemir & Sahal, 2018). For instance, Ozdemir and Sahal (2018) used two prompts: one asking students to pose a problem similar to a given example and another asking them to pose a problem based on a real-life story or data. Although different in format, both prompts fall under the category of generating new problems rather than reformulating existing ones. In contrast, Chen et al. (2015) and English (1997) used prompts representing both types: generating new problems and reformulating given ones. In Chen et al.'s study, students participated in an 11-unit PP-integrated course designed to improve PP and problem-solving competence, attitudes, and mathematical achievement. Students were introduced to various posing strategies and evaluation criteria through a written guide—a “PP instruction card” (see Figure 5 for details)—which included examples of PP types (e.g., generating from situations, transforming given problems), strategies (e.g., reversing knowns and unknowns, adding constraints), and evaluation rubrics (e.g., solvability, originality, clarity, complexity). Each 45-minute session followed a structured format: an introduction, small-group PP, solving peers' problems, individual PP, and a whole-class reflection. Findings showed that students in the P-PBL group made significant gains in PP originality from pre- to post-test, although no improvements were observed in appropriateness, complexity, or diversity. In terms of problem-solving competence, P-PBL students also outperformed their peers. Although students' attitudes and beliefs about PP and problem solving improved significantly, no significant between-group differences were found in pre- or post-test comparisons.

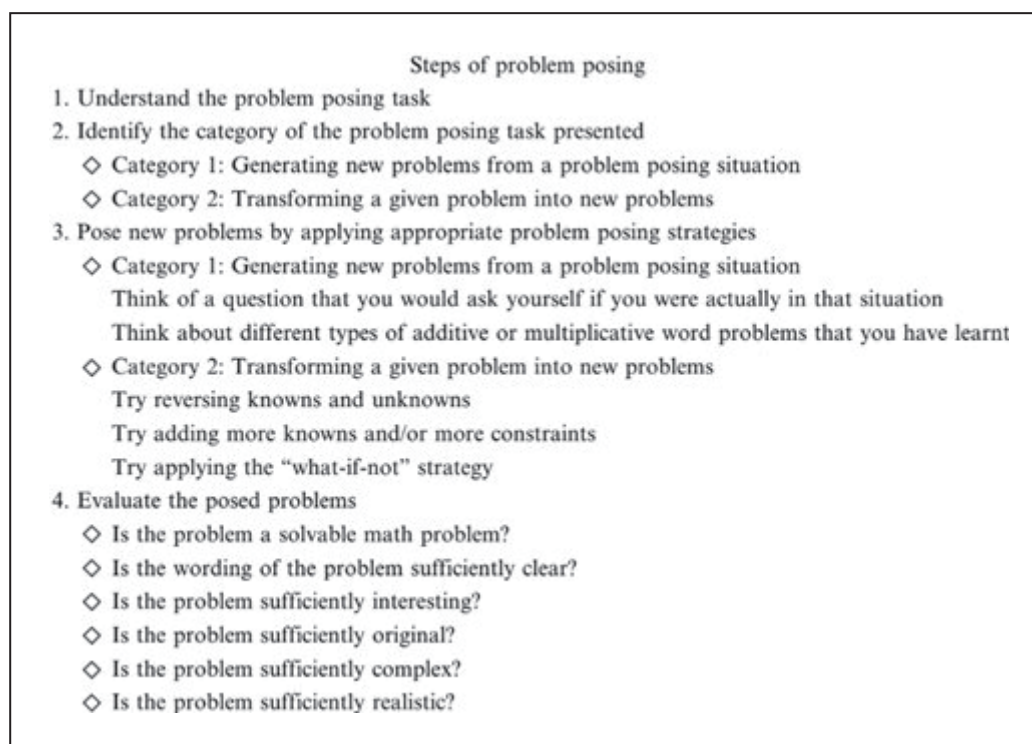


Figure 5. PP instruction card (Chen et al., 2015)

English (1997) employed a broad range of PP prompts in her study, including semi-structured prompts for generating new problems and structured prompts for reformulating existing ones. She examined elementary students’ perceptions and preferences for different problem types, their recognition and use of problem structures, and the development of their mathematical thinking through a 10-week P-PBL-integrated course. Throughout the course, students engaged in various PP activities to deepen their understanding of problem structures. Early in the program (Weeks 3–5), students compared routine and nonroutine problems, identified similarities and differences, and posed problems with similar structures. In subsequent sessions, students discussed strategies for creating new problems under varying conditions, such as being given only knowns, only constraints, or both. Later, they practiced modifying given problems by applying specific techniques like reversing knowns and unknowns, removing or adding constraints, and using the “what-if-not” strategy. All PP tasks were completed individually. The findings showed a significant increase in students’ PP competence, particularly in generating problems that were structurally complex and diverse. The findings revealed an increase in the range of problems students expressed interest in solving from pre- to post-test. Notably, students often described “real world” problems as more challenging, commenting that such problems “have lots of things to work out.” The P-PBL-integrated course was also found to positively impact students’ understanding of problem structures. Additionally, there was a significant improvement in students’ PP competence, particularly in their ability to create mathematically rich problems that were more structurally complex and diverse.

Settings for Students’ Problem Posing

The third step in Cai’s (2022) P-PBL instructional model focuses on students’ actual PP. In reviewing P-PBL studies, we examined how students were asked to pose problems individually, in pairs, in small groups, or as a whole class. Figure 6 presents the distribution of these settings. The most common approach was individual PP (18 studies), followed by small groups (4 studies), free choice (3 studies), pairs (2 studies), both individual and pair formats (2 studies), whole class (1 study), and not reported (1 study). Although individual PP dominates, several studies implemented more unique settings that offer essential student engagement and learning insights.

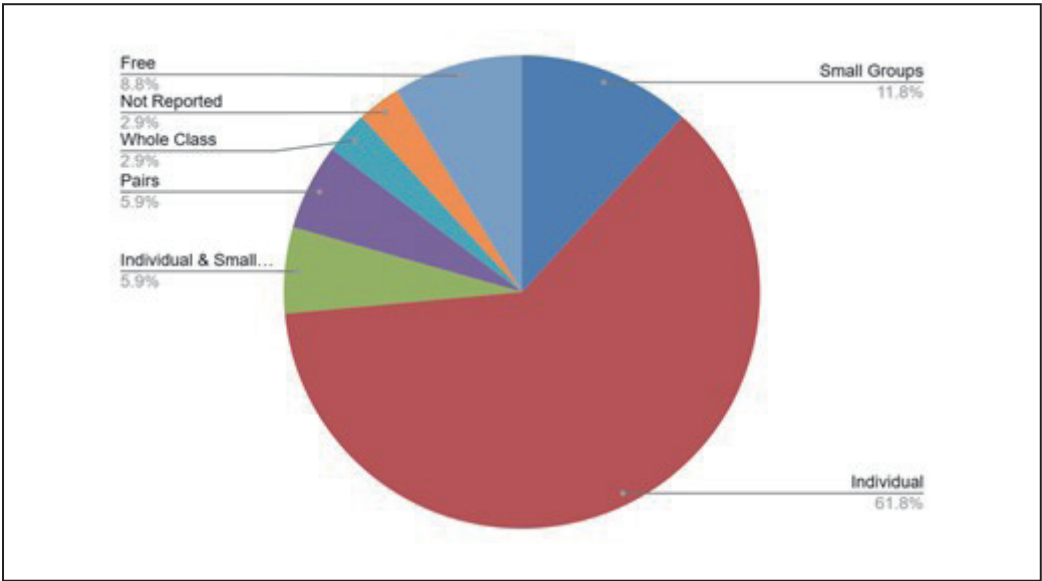


Figure 6. Distribution of types of students’ PP settings

One particularly unique approach involved students posing problems collaboratively as a whole class. Cankoy (2014) examined the effects of a collaborative PP strategy on fifth graders' PP performance in the only identified study using this method. In this study, the teacher began by writing a few words on the board, and students took turns adding to the sentence until they collectively created a solvable problem. If the problem was unsatisfactory, students revised it in small groups. The collaborative PP group produced more solvable and realistic problems than those who posed individually and generated more start-unknown problem types (e.g., "The sum of Jeff and his father's age is 56. How old is Jeff if his father is 26 years older?"). The study concluded that whole-class PP fostered deeper cognitive engagement.

Another innovative setting paired students with student teachers (STs) in a one-on-one PP environment. Lowrie (2002) studied this approach with 25 first-grade and 28 third-grade students, each paired with an ST for five weekly 1-hour sessions. Students followed a four-phase P-PBL model: posing a problem, planning a solution, solving the problem, and reflecting on the process. In the first session, students responded to the prompt "Write (or tell me about) a mathematics problem you would like to solve" (Lowrie, 2002, p. 90) and discussed their ideas with their STs, who asked guiding questions to probe their thinking. Over time, STs increasingly supported students in refining their PP skills. The results showed that students could pose one- and multi-step problems, explain the mathematical thinking behind them, and progressively generate more sophisticated problems across the sessions.

Some studies also explored technology-enhanced and game-based PP environments. Chang et al. (2012) examined the effects of a system-based, game-supported PP platform compared to a paper-based model. Their design followed Leung's (1993) adaptation of Polya's four-step model: posing, planning, solving, and looking back. In the system-based model, fifth-grade students created problems and entered them with correct answers into the platform. The system allowed students to verify and revise their problems if they wanted. It also allowed the students to solve the problems posed by themselves or others and to receive feedback on performance (e.g., accuracy, time, or scores). This interactive loop helped students generate new problems more effectively. Findings showed that the system-based model benefited most low-performing students in problem solving. Additionally, students in the system-based group outperformed the paper-based group in flexibility, originality, accuracy, and elaboration in PP. Their flow experience—engagement and immersion—was also significantly higher in the system-supported P-PBL environment. Together, these studies demonstrate that although individual PP remains the norm, alternative settings (e.g., whole-class collaboration, one-on-one student–teacher interactions, and digital platforms) can offer rich, engaging opportunities for students to develop and refine their problem-solving competencies.

Handling of Posed Problems in P-PBL Studies

Among the 31 P-PBL studies reviewed, none focused exclusively on the effects or detailed processes of how teachers handled students' posed problems, but only four studies provided some insights into what was done with the problems after students posed them. The first two—conducted by Winograd (1993) and Watson and English (2017)—highlighted the iterative nature of problem creation and revision, emphasizing refinement over time. The other two studies, by Manuel and Freeman (2017) and Peng et al. (2020), documented how students posed problems for others to solve, whether peers in the same classroom, another classroom, or even in a different country. These studies extended Step 3 of Cai's P-PBL model (students pose problems), illustrating how posed problems were used in meaningful follow-up activities, including problem solving, peer feedback, and cross-classroom engagement. Rather than ending with creating problems, these studies engaged

students in using and reflecting on the problems they posed, deepening their mathematical thinking and the authenticity of the learning experience.

Winograd (1993) investigated fifth-grade students' processes of writing mathematical story problems as part of a long-term P-PBL implementation. Over 5 months, students from the same classroom engaged in PP activities three to four times weekly. In each session, students were asked to pose story problems grounded in their real-life experiences, aligned with a mathematical topic selected by the teacher. The P-PBL implementation had two distinct components. In the first phase, after individually writing their problems, students were invited to present them to the class by writing them on the board. Classmates were then invited to solve the posed problems and share their solutions. Whole-class discussions followed, often led by the poser or classmates, focusing on various solution strategies and mathematical reasoning. The teacher supported these discussions by revisiting key strategies and relevant mathematical concepts. Students were also encouraged to give feedback on the posed problems, comment on aspects they liked, or offer suggestions for revision. In the second phase, students posed their problems to peers in small groups. Each student shared their problem with another group, who attempted to solve it. To conclude the lesson, selected students shared insights from small-group discussions with the entire class. Although the study identified three specific strategies students used to write story problems, those findings are beyond the scope of this review and will be discussed elsewhere.

Watson and English (2017) examined sixth-grade students' PP competence, focusing on their ability to formulate and refine statistical questions based on data. The study was part of a 3-year longitudinal project investigating how upper elementary students develop statistical problem-solving and informal inference skills as a foundation for statistical literacy. Students engaged in a four-step decision-making process: (1) pose questions, (2) collect data, (3) analyze data, and (4) make decisions. The activity began with students viewing video clips of Usain Bolt's Olympic performances (2008 and 2012) followed by an intentionally vague prompt: "Are athletes getting better over time?" In small groups and whole-class discussions, students reflected on the vagueness of the question and what specific measures or data would be needed to answer it meaningfully. Students were then asked to individually pose refined statistical questions, identifying relevant data that could support their inquiry. After peer discussions about each other's questions and potential challenges, students worked in pairs to select from 12 available data sets and further revised their questions. They also created hand-drawn sketches outlining how to analyze the data to answer their refined questions. The students received TinkerPlots files and used the software to analyze data and test their conclusions. They then watched a video about how technological advancements have influenced athletic performance, and some final reflections on their findings were prompted. The study found that approximately 70% of students were fully engaged in the statistical PP cycle—from posing and refining questions to analyzing data, drawing conclusions, and re-evaluating their interpretations in light of new information. These students demonstrated the ability to build coherent arguments, revise questions based on data constraints, and reflect critically on their conclusions.

Peng et al. (2020) investigated the similarities and differences in the characteristics of mathematical problems posed by students in China and Canada and the strategies students used to solve problems posed by their international peers. The study involved 40 third-grade students—20 from a Chinese elementary school and 20 from a Canadian elementary school. Students in both classes posed problems either individually or in small groups. After presenting their problems to classmates and their teacher, each class voted on their favorites: the Chinese class selected six problems, and the Canadian class selected four. These selected problems were then exchanged between the two classes using Skype. Using Skype meetings, the two classes presented their selected mathematical problems

to their counterparts, and each class received the problems and was asked to solve them as “game players.” The classes switch their roles between “game play manager” and “game player” after each mathematical problem. The Chinese students’ problems spanned four content areas—including geometry, spatial sense, and algebra—and half were embedded in real-life contexts (e.g., baking, paper cutting). In contrast, all four problems from the Canadian students focused on number sense and were not tied to real-world scenarios. When solving their peers’ problems, the Canadian students worked collaboratively, applied various strategies, and offered feedback after explaining their solutions. In contrast, Chinese students solved all four Canadian problems individually, relying on formulas or models, and focused on clearly presenting their solutions rather than providing feedback. This study illustrates how posed problems served as the foundation for cross-classroom mathematical exchange and discussion, highlighting cultural and pedagogical differences in how PP and problem solving were approached and enacted.

The final study, conducted by Manuel and Freiman (2017), also documented students posing problems for others to solve. In addition to exploring this “posing for others” aspect, the study examined how students’ participation in whole-class discussions about what makes a rich mathematical problem, along with the integration of technology (audio/video), influenced both the nature of the problems they posed and their perceptions of the P-PBL experience. Students were tasked with creating math problems for a shared website accessible to students from other schools. Before posing, they participated in three preparatory workshops: (1) an overview of the website’s design and functionality, (2) a session on how submitted solutions would be assessed and feedback delivered, and (3) a whole-class discussion on identifying characteristics of rich mathematical problems. Following the workshops, students worked in small groups to create three sets of problems without restrictions on content area or problem type. Throughout the process, students had opportunities to discuss and refine their problems with teachers and researchers, aiming to increase their richness and depth. A total of 23 problems were posted to the website. Once published, student authors received data on how their problems were solved, including submitted solutions, analyses, and peer feedback. They also had opportunities to review these solutions and provide formative feedback to the solvers. The study found that most posed problems were moderately complex, often requiring multiple steps to solve, though they tended to resemble problems students had encountered in class. Many students preferred continuing with PP activities in future years, noting that posing problems felt more challenging than solving them.

Discussion

Recommendations for Designing and Implementing P-PBL

As part of this systematic review of P-PBL literature, we examined how each component of the P-PBL instructional model was implemented across the studies. Drawing on these findings and insights from recent meta-analyses of PP interventions, we offer the following recommendations for researchers, teachers, and teacher educators to maximize the instructional potential and student engagement in mathematics through a series of thoughtfully designed PP tasks.

Step 1: Presenting Problem Situations

We recommend diversifying the types of problem situations used in P-PBL instruction. Although mathematical expressions can support students in focusing directly on mathematical content, their simplicity may not be sufficient for developing students’ PP competence, especially for students who struggle with translating symbolic or numerical information into word problems. In such cas-

es, annotated mathematical expressions add contextual clues and can help scaffold students' formulation of meaningful problems (Cheng et al., 2014; English, 1997). Alternatively, suppose the goal is to assess students' mathematical translation or comprehension skills. In that case, unannotated expressions may serve better by requiring students to interpret the numerical content and convert it into coherent verbal problems. Real-life contexts also play a vital role in P-PBL. They allow students to draw on their personal experiences. Artifacts such as amusement park brochures or grocery flyers have proven particularly effective, especially for students with learning difficulties or lower prior achievement (Bicer et al., 2020; Bonotto, 2013). Images, stories, or everyday artifacts can serve as valuable problem situations that help students pose problems by building on their familiar everyday knowledge.

Step 2: Presenting Problem-Posing Prompts

We recommend the proper use of specific PP prompts with specific pedagogical goals. For example, prompts that ask students to create “similar” problems can help them better understand problem structure (Palmér & van Bommel, 2020). Students can identify structural similarities and differences between their posed problems and given ones, which often have similar solution strategies. On the other hand, some less constrained prompts, such as “pose as many problems as possible based on the given situation,” can be powerful tools for assessing and developing students' creativity (Bicer et al., 2020). By allowing varied responses, these prompts will promote students' mathematical thinking, hopefully creative thinking.

Step 3: Setting for Student Problem Posing

We recommend utilizing both individual PP and group PP in P-PBL implementation. For instance, students could begin by creating their problems independently, followed by discussions in pairs or small groups. Having students pose problems first encourages them to draw on their mathematical understanding and supports individual cognitive engagement. Afterward, sharing and discussing these problems in pairs or small groups can enhance their understanding of the same problem situation, promote productive mathematical discourse, and expose them to various approaches and ideas.

Purposeful student pairing is also valuable. Pairing students with peers or mentors who can provide in-the-moment feedback helps sustain focus on the intended mathematical goals and improves the sophistication of students' posed problems (Lowrie, 2002). In this role, a knowledgeable partner or teacher can prompt students to reflect, clarify, and revise their thinking. Additional settings, such as game-based PP systems, can motivate students, particularly lower performing or less intrinsically motivated ones, by adding a layer of external engagement (Chang et al., 2012). Whole-class PP can also foster collaboration and shared problem ownership; however, it is most effective when preceded by individual PP so that students can formulate their ideas before co-constructing problems (Cankoy, 2014).

Step 4: Handling Posed Problems

We recommend that P-PBL instruction include follow-up activities involving posed problems rather than treating PP as an isolated task. For example, pairing PP with problem solving supports deeper conceptual understanding (Kapur, 2015, 2018). Without such follow-up, students may not reflect critically on their work or develop the skills to pose rich, solvable problems. Moreover, creating problems for an authentic audience—whether classmates, students in other schools, or even international peers—encourages students to think more metacognitively about the kinds of problems they pose and how others will interpret them (Peng et al., 2020). Online platforms and classroom web-

sites can facilitate this process by enabling students to share problems, receive feedback, and reflect on others' solutions (Manuel & Freiman, 2017).

We also emphasize engaging students in complete PP cycles, including revising, solving, and analyzing posed problems. Opportunities to revise problems based on feedback—from peers or teachers—support growth in PP competence by helping students attend to solvability, difficulty, and topical relevance (Watson & English, 2017; Winograd, 1993). When such cycles are adequately implemented, they can lead to some improvement in students' PP competence and to gains in overall mathematical achievement (Ozdemir & Sahal, 2018).

Conclusion

In the present study, we offered several recommendations based on the existing research on P-PBL. As discussed earlier, our understanding remains limited regarding how various P-PBL components—such as the problem situation, PP prompts, student settings for posing problems, and teachers' handling of student-generated problems—shape the enactment of P-PBL and influence student learning outcomes (e.g., PP competence, problem solving, creativity, attitudes, and self-efficacy). Given that P-PBL remains a relatively new instructional approach in mathematics education, further empirical investigation is urged to explore how it can serve as an instructional strategy to enhance students' mathematical learning in meaningful and engaging ways. As highlighted previously, there is a notable gap in research examining how teachers respond to and manage student-posed problems. Although some studies have begun to document classroom implementation of P-PBL (e.g., Zhang & Cai, 2021), we still have a limited grasp of how teachers make instructional decisions, particularly regarding selecting and using student-generated problems to guide mathematical discussions toward intended learning goals (Cai et al., 2023).

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Invited Lecture 3

**THE IMPACT OF SELF-REGULATED LEARNING ON
MATHEMATICS PERFORMANCE:
INSIGHTS FROM EAST ASIAN STUDENTS IN PISA 2022**

Zhu Yan

**CREATING MATHEMATICS CLASSROOMS WHERE ALL STUDENTS CAN
SUCCEED AND ENJOY LEARNING**

JinHo Kim

**MATH IS FUN WITH BUNDLE-NUMBERS ON A
BUNDLE-BUNDLE-BOARD**

Allan Tarp



**The 9th ICMI-East Asia Regional Conference on
Mathematics Education**



THE IMPACT OF SELF-REGULATED LEARNING ON MATHEMATICS PERFORMANCE: INSIGHTS FROM EAST ASIAN STUDENTS IN PISA 2022

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Self-regulated learning (SRL) has been recognized as a critical factor influencing academic achievement. This study leverages PISA 2022 data to explore the relationship between SRL behaviors and mathematics performance, with a particular emphasis on comparing East Asian education systems to the OECD average. The results indicate that, despite engaging in fewer SRL behaviors—such as posing questions and participating in group discussions—students from East Asia consistently outperform their OECD peers in mathematics. Additionally, the analysis reveals significant regional variations within East Asia. Singapore reported the highest proportion of students engaging in question-asking behaviors, while Chinese Taipei had the lowest. In contrast, South Korean students demonstrated strong proactive learning behaviors, such as completing assignments early and integrating new knowledge with prior learning, which were positively correlated with enhanced mathematics performance. Interestingly, high-performing systems like Chinese Taipei and Macao achieved impressive results despite relatively lower levels of SRL engagement, suggesting that external instructional factors play a crucial role in student success. These findings underscore the complex interaction between individual learning behaviors and systemic influences, indicating that policymakers should design SRL strategies tailored to the unique strengths and educational contexts of each system. Such strategies should also account for cultural and instructional differences to optimize student performance.

Keywords: Self-regulated learning (SRL), Mathematics performance, PISA 2022, East Asia, Educational policy

Introduction

Self-regulated learning (SRL) constitutes a multidimensional, dynamic process whereby learners actively engage in setting academic goals, selecting and implementing appropriate cognitive and metacognitive strategies, monitoring their progress, and adjusting their efforts in response to ongoing evaluations of performance. This process also entails the regulation of motivational beliefs and behavioral engagement, thereby enabling learners to become autonomous and effective agents of their own learning (Zimmerman, 2002). Over the past several decades, SRL has become a cornerstone of contemporary educational psychology, with a vast body of empirical research affirming its pivotal role in fostering academic engagement, resilience, and achievement across a variety of domains, including mathematics education (Hudesman et al., 2011; Wang & Sperling, 2020).

Specifically, interventions that scaffold SRL processes—incorporating mechanisms such as strategic goal-setting, deliberate planning, self-monitoring, reflective self-evaluation, and motivational regulation—have been shown to significantly enhance both learning outcomes and learner autonomy (Blalock et al., 2017; Schunk & DiBenedetto, 2020). The effectiveness of such interventions is further amplified by the contribution of executive functions, such as cognitive flexibility, inhibitory control, and working memory capacity, which facilitate the management of complex problem-solving tasks, especially in mathematically intensive contexts (Rutherford et al., 2018). These findings suggest that SRL is not only a set of techniques or behaviors but a holistic regulatory system that supports lifelong learning and adaptability.

Despite the widespread validation of SRL in Western educational contexts, a growing body of literature highlights the pressing need to explore its cross-cultural applicability, particularly within East Asian educational systems. These systems—exemplified by countries such as China, Japan, and South Korea—are internationally recognized for their high levels of academic performance, yet are deeply embedded in sociocultural traditions that diverge significantly from Western norms. While East Asian students often exhibit sophisticated metacognitive skills, high levels of self-efficacy, and strong academic discipline, their engagement with SRL strategies is mediated by cultural constructs such as collectivism, Confucian values, and hierarchical social structures (Chiu, 2017).

In collectivist cultures, educational behaviors are often shaped by the imperatives of social harmony, deference to authority, and familial or communal expectations. As a result, SRL strategies commonly promoted in Western contexts—such as help-seeking, critical self-reflection, or autonomous goal-setting—may be constrained or reframed within East Asian paradigms that emphasize perseverance, conformity, and social obligation (Thornton et al., 2012). For instance, reluctance to seek help may stem not from lack of skill but from concerns about “losing face” or burdening others, thereby complicating conventional interpretations of SRL effectiveness across cultural lines. These cultural nuances underscore the limitations of applying Western-centric SRL frameworks universally and point to the necessity of culturally responsive models that account for localized epistemologies, motivational orientations, and socio-educational pressures.

The present study seeks to interrogate the relationship between SRL behaviors and mathematics achievement among East Asian students, leveraging data from the 2022 cycle of the Programme for International Student Assessment (PISA). Two central research questions structure this inquiry: (1) What is the nature and strength of the association between SRL strategies and mathematics performance in East Asian educational settings? and (2) How do these patterns align with or diverge from trends observed across the broader cohort of OECD countries? While PISA does not provide direct measures of cultural variables, the study employs contextual analysis to interpret SRL behaviors through the lens of systemic and cultural forces—such as exam-oriented educational policies, teacher-centered pedagogies, and pressure for high-stakes performance—that may either enable or inhibit the enactment of SRL.

In advancing both theoretical and applied understanding, this research contributes to the refinement of SRL models by foregrounding the role of sociocultural context. Its findings offer critical insights for educators, curriculum designers, and policymakers seeking to cultivate self-regulated learning in ways that are culturally consonant and contextually relevant. By illuminating the intersections of SRL with cultural values, institutional expectations, and learner motivation, this study aims to inform the design of pedagogical interventions that are both effective and equitable. Ultimately, it advocates for a more globally inclusive approach to SRL scholarship—one that respects cultural diversity while fostering universal principles of learner agency, critical thinking, and lifelong educational engagement.

Literature Review

Self-Regulated Learning and Academic Achievement

Self-regulated learning (SRL) encompasses a comprehensive, cyclical process wherein learners exert proactive control over their cognitive, motivational, and behavioral engagement in academic tasks. Central to this framework is the orchestration of goal-setting, strategic planning, self-monitoring, self-evaluation, and the regulation of affect and motivation (Schunk & DiBenedetto, 2020). This multifaceted process enables learners to approach academic tasks with increased agency and

adaptability, particularly in complex disciplines such as mathematics, where sustained attention, conceptual reasoning, and iterative problem-solving are paramount (Li et al., 2018; Rosário et al., 2013).

The empirical foundation supporting SRL's impact on academic achievement is robust and consistent. Studies have demonstrated that learners who effectively deploy SRL strategies are more likely to persist through academic challenges, engage in deeper learning, and exhibit heightened performance, particularly in domains that demand sustained effort and autonomous thinking. For instance, Lai and Hwang (2021) identified metacognitive strategies—especially planning and self-monitoring—as significant predictors of success in digital learning environments for mathematics. In such contexts, where learners operate with elevated autonomy, the ability to regulate learning becomes a critical determinant of achievement.

Moreover, SRL is tightly interwoven with motivational constructs, such as self-efficacy, task value, and intrinsic interest. These beliefs not only fuel sustained engagement but also reinforce learners' confidence in their capacity to succeed, thereby contributing to positive feedback loops that enhance effort and perseverance (Muwonge et al., 2019). In flipped classroom settings, where students are expected to prepare independently before engaging in in-class collaborative work, the capacity to self-regulate—particularly in managing time and reflecting on performance—has been shown to strongly correlate with learning gains (Sun et al., 2018). These findings affirm that SRL is not a peripheral skill but a central mechanism through which learners achieve academic success, especially in learner-centered or inquiry-based environments.

However, despite its theoretical and practical importance, the development of SRL competencies can be constrained by systemic and contextual barriers. Limited access to formative feedback, insufficient scaffolding, and rigid curricular structures often undermine opportunities for learners to internalize SRL practices (Seban & Urban, 2024; Vishwakarma & Tyagi, 2023). As such, educational interventions must be designed with contextual sensitivity, ensuring that strategies for promoting SRL are responsive to students' diverse sociocultural realities, technological environments, and emotional needs. Tailored approaches are especially vital in educational systems characterized by high-stakes testing and traditional pedagogy, where SRL must be explicitly taught and reinforced rather than implicitly assumed.

PISA Framework and its Relevance to SRL

The Programme for International Student Assessment (PISA) provides a globally standardized framework for evaluating students' competencies in reading, science, and mathematics, with an emphasis on the application of knowledge to real-world contexts. PISA's assessment model aligns closely with core SRL processes, particularly in its focus on problem-solving, critical thinking, and metacognitive regulation (Stacey, 2015). Rather than merely testing procedural fluency or rote memorization, PISA tasks require students to engage in goal-oriented reasoning, monitor their progress across multiple steps, and adjust their strategies in response to feedback and contextual cues—hallmarks of effective self-regulation (Kramarski & Revach, 2009).

Beyond individual assessments, PISA's comprehensive data collection facilitates macro-level insights into how educational systems support or hinder SRL development. Comparative studies, such as Liang's (2010) analysis of high-performing systems in Finland and Canada, reveal that nations scoring consistently well on PISA often integrate SRL-supportive pedagogical practices—such as student reflection, incremental goal-setting, and formative assessment—within their national curricula. These systems emphasize teacher facilitation over direct instruction, encouraging students to take ownership of their learning and apply metacognitive strategies across disciplines.

Furthermore, PISA's international scope allows for culturally informed analysis of how students approach novel, unstructured problems. Such tasks are particularly revealing of students' metacognitive awareness and strategic adaptability, making PISA a valuable tool for assessing SRL across diverse educational contexts. The design of the PISA framework enables researchers to investigate how systemic variables—such as curriculum design, instructional practices, and assessment regimes—influence students' engagement with SRL. This is particularly relevant when considering the ways cultural values, such as individualism in Western countries versus collectivism in East Asia, shape students' learning orientations and regulatory strategies.

Through its inclusion of attitudinal and contextual questionnaires, PISA also provides insights into students' learning beliefs and environments, offering rich correlational data on the factors that promote or inhibit self-regulation. For researchers examining the intersection of culture and cognition, PISA thus represents not only a diagnostic tool but also a springboard for theorizing how SRL manifests differently across educational ecosystems.

Cultural Dimensions of SRL in East Asian Educational Contexts

Educational systems in East Asia—particularly those in countries such as China, Japan, South Korea, Singapore, and Chinese Taipei—are frequently lauded for their exceptional performance in international assessments like PISA. These outcomes are often attributed to culturally embedded educational values emphasizing discipline, diligence, and collective responsibility (Tan, 2018). However, the implementation and expression of SRL in these contexts diverge from Western paradigms due to the influence of Confucian heritage cultures, which prioritize effort over innate ability, respect for authority, and social harmony (Chen, 2014; Leung & Revina, 2023).

While teacher-centered instruction remains prevalent in many East Asian classrooms, it does not preclude the development of SRL; rather, it shapes its form. For instance, the internalization of self-monitoring behaviors is often promoted through structured repetition and mastery learning, while motivational regulation is encouraged through familial and societal expectations of success. In these systems, self-regulation is not solely an individual endeavor but is reinforced through peer comparisons, parental involvement, and collective classroom accountability. This collectivist framing aligns with the notion that SRL can be both individually and socially mediated, challenging Western assumptions that equate self-regulation with autonomy and independence.

Recent educational reforms across East Asia reflect a growing awareness of the need to cultivate SRL in more explicit and student-centered ways. Singapore, for example, has restructured its curriculum to include inquiry-based learning, interdisciplinary projects, and continuous assessment, all of which aim to foster student autonomy within a supportive framework (Ee & Seng, 2008). Similarly, Hong Kong has experimented with hybrid pedagogies that merge Western constructivist practices with traditional East Asian rigor, thereby promoting SRL in the form of guided problem-solving, reflection journals, and collaborative learning (Ho, 2017).

Nevertheless, the cultural and systemic pressures that pervade these environments can also act as impediments to SRL. In Shanghai, for instance, the overwhelming emphasis on standardized testing and school rankings often leads to extrinsic motivation, performance anxiety, and surface-level learning strategies, which may curtail the development of deeper SRL capacities (Swierczek & Bechter, 2010). Conversely, Chinese Taipei has emphasized collaborative learning and peer dialogue, which supports the development of SRL strategies that align with both intrinsic motivation and communal values (Tan, 2018). These divergent trajectories illustrate the complex interplay between educational policy, cultural values, and learner psychology in shaping SRL outcomes.

In synthesizing these insights, it becomes evident that SRL in East Asia cannot be fully understood without attending to the sociocultural, institutional, and policy contexts in which learning occurs. Culturally responsive approaches are therefore critical—both in research and practice—to ensure that SRL interventions resonate with students’ lived realities, aspirations, and cultural identities. By accounting for such variability, educators and policymakers can better support the integration of SRL into curricula in ways that honor local traditions while promoting globally relevant competencies.

Theoretical Framework

Self-Regulated Learning in a Socio-Cultural Context

Self-regulated learning (SRL) has emerged as a foundational construct in understanding mathematics achievement, given the discipline’s inherent demand for strategic thinking, metacognitive control, and adaptive problem-solving. The theoretical basis for SRL is grounded in Zimmerman’s (2000) cyclical model, which delineates three interrelated phases: the forethought phase, where learners engage in goal setting and mobilize motivational beliefs (e.g., self-efficacy); the performance phase, involving the deployment of cognitive and metacognitive strategies along with ongoing self-monitoring; and the self-reflection phase, wherein students evaluate their performance and adjust future learning behaviors. This recursive process enables learners to iteratively refine their approaches, thereby enhancing mathematical resilience and strategic flexibility in tackling increasingly complex tasks (Chen et al., 2018).

Motivation operates as a central mechanism within this framework, particularly through the lens of self-efficacy—a learner’s belief in their capability to execute actions required for success. Schunk’s (1989) seminal work elucidates the powerful influence of self-efficacy on the selection, initiation, and persistence of learning strategies. Students with high self-efficacy are more likely to persevere in the face of difficulty, exhibit strategic initiative, and engage in deeper metacognitive reflection. Conversely, low self-efficacy correlates with avoidance behaviors, diminished effort, and a reliance on extrinsic cues, often exacerbating underachievement (Bol et al., 2015; Fauzi & Widjajanti, 2018). Importantly, intrinsic motivation—defined as engagement driven by inherent interest and enjoyment—also plays a critical role in sustaining SRL behaviors over time. Learners who find intrinsic value in mathematics are more inclined to self-monitor, reflect meaningfully, and persist in solving non-routine problems.

The application of SRL principles, however, is not culturally neutral. In East Asian educational contexts, pedagogical traditions and sociocultural norms can mediate the extent to which SRL strategies are fostered or constrained. Instruction in these settings tends to emphasize teacher-centered delivery and extrinsic motivators, such as social comparison and exam performance (El-Adl & Alkharusi, 2020). While these practices can cultivate high levels of procedural fluency and academic discipline, they may also limit student autonomy and suppress the development of metacognitive competencies by reinforcing a top-down learning dynamic (Yıldızlı & Saban, 2016).

This cultural-emphasis on structure and external validation raises critical questions about how best to integrate SRL-enhancing strategies within traditionally didactic systems. Scholars have advocated for the intentional embedding of SRL-supportive interventions into structured curricula, including goal-setting scaffolds, strategy modeling, metacognitive prompts, and reflective journaling (Hudesman et al., 2011; Sun et al., 2018). These interventions not only bolster strategic awareness but also reframe learning as a personally meaningful process. Rutherford et al. (2018) underscore the importance of designing such interventions in ways that cultivate autonomous motivation, en-

abling students to internalize SRL as a process of self-growth rather than mere task compliance. The promotion of reflective autonomy within a culturally resonant structure may serve as a promising approach to reconciling SRL with East Asian pedagogical traditions.

In sum, the SRL framework offers a multidimensional lens through which mathematics learning can be optimized, encompassing cognitive, metacognitive, and motivational dimensions. Yet, its effective implementation must be sensitive to cultural contexts and educational norms. A more nuanced, culturally responsive adaptation of SRL—one that acknowledges the interplay of motivation, structure, and autonomy—has the potential to foster not only immediate academic success but also lifelong learning capacities.

Cultural and Instructional Contexts Impacting SRL in East Asia

Educational systems in East Asia are deeply embedded in collectivist cultural traditions that prioritize social harmony, hierarchical respect, and communal success. Within this cultural matrix, academic achievement is not solely viewed as a personal accomplishment but rather as a fulfillment of familial and societal expectations (Tan et al., 2021). Consequently, SRL strategies in such contexts often serve instrumental rather than intrinsic purposes. In South Korea, for instance, students frequently adopt self-regulatory behaviors—such as time management and rote memorization—not out of personal interest, but to meet rigid societal standards, including university entrance exams and familial aspirations (Jang et al., 2023). While these behaviors can enhance academic outcomes in the short term, they risk undermining deeper engagement and the development of autonomous learning dispositions.

The instructional architecture of East Asian classrooms further complicates the integration of SRL principles. Traditional pedagogies in countries like China, Japan, and South Korea emphasize teacher authority, didactic instruction, and content mastery delivered through highly structured curricula (Peeters et al., 2016; Cheng et al., 2023). In such environments, students are frequently positioned as passive recipients rather than active agents of knowledge construction. “Self-regulation,” under this model, often becomes synonymous with compliance—the ability to follow instructions meticulously and fulfill externally defined standards. While this promotes procedural accuracy and disciplined study habits, it also tends to marginalize reflective autonomy and higher-order metacognition.

Nevertheless, recent educational reforms across East Asia suggest a growing recognition of the limitations inherent in overly rigid pedagogical models. Singapore and South Korea, for example, have begun to pilot hybrid instructional approaches that integrate scaffolded autonomy within structured environments. These models encourage students to gradually assume more control over their learning processes, especially in problem-based tasks, while retaining the supportive presence of teacher guidance. Reflective dialogues, personalized feedback, and strategic questioning are increasingly employed to help students connect learning strategies with academic outcomes (Jang et al., 2023). Furthermore, formative assessment practices that prioritize growth over rankings are being introduced to shift the motivational emphasis from extrinsic validation toward personal mastery.

Despite these promising developments, deep-rooted cultural and systemic barriers continue to hinder the widespread adoption of student-centered pedagogies. Educational stakeholders may resist reforms perceived as diluting academic rigor or disrupting traditional hierarchies. The success of SRL-oriented reforms, therefore, hinges not only on instructional innovation but also on a broader cultural negotiation—one that balances respect for communal values with the imperative of cultivating independent, reflective learners.

In conclusion, the interplay between cultural collectivism and instructional norms in East Asia presents a complex landscape for SRL implementation. While extrinsic motivation and structured environments contribute to high academic achievement, they often do so at the expense of metacognitive autonomy and intrinsic engagement. A contextually responsive approach—integrating SRL principles within existing structures without disregarding cultural values—offers a promising path forward. By embedding scaffolded SRL interventions within teacher-directed instruction, educators can help students excel not only in procedural domains but also in unstructured, higher-order problem-solving tasks. This balance between tradition and innovation is crucial for nurturing resilient, self-directed learners equipped to navigate the demands of 21st-century education.

Research Methods

Data Source

This study analyzes data from the mathematics-focused cycle of PISA 2022, focusing on six high-performing East Asian economies: Singapore, South Korea, Japan, Hong Kong (China), Macao (China), and Chinese Taipei. These regions collectively represent nearly 35,000 sampled 15-year-olds, drawn from a global pool of 690,000 students across 81 economies (see Table 1). Notably, Singapore, Hong Kong, and Macao exclusively sampled students in academic programs, while Japan, South Korea, and Chinese Taipei included both academic and vocational tracks. This distinction is critical, as the inclusion of both educational pathways reflects systemic diversity in the East Asian region, which could potentially influence SRL behaviors and mathematics performance outcomes.

The sampling design ensures diverse representation, with gender distribution balanced (47.9%–50.4% female) and socioeconomic and cultural diversity further ensuring robust representation of these East Asian educational contexts. The two-stage stratified random sampling design was employed, first selecting schools as primary units, followed by random student sampling within these schools (Adams & Wu, 2002; Ma et al., 2008). This method ensures the representativeness of the sample and allows for an analysis that captures the diversity across different educational systems and socio-cultural contexts, contributing to a more comprehensive understanding of educational outcomes in these economies.

Table 1. PISA 2022 target populations and samples in six East Asian economies

Economies	15-year-olds population	Sampled students
Japan	1,109,590	5,760
Korea	418,028	6,454
Hong Kong (China)	59,241	5,907
Macao (China)	4,500	4,384
Singapore	44,037	6,606
Chinese Taipei	205,632	5,857

Measures and Variables

PISA 2022 provides eight sets of ten plausible values (PVs) to assess mathematics performance, including one set for *overall mathematics achievement* and four content-specific domains (Change and Relationships, Quantity, Space and Shape, Uncertainty and Data), along with three process-specific domains (Employ, Formulate, Interpret). For this study, only the overall mathematics plausible values (PV1MATH–PV10MATH) were analyzed, as the investigation focuses on general achievement rather than domain-specific competencies.

This study operationalizes two core SRL constructs, both of which align with PISA 2022’s framework for sustainable lifelong learning. These constructs were selected to provide valuable insights into students’ ability to manage their learning processes and persist in mathematics tasks, which are critical for both academic success and preparing students for future global challenges. One is *learning control and self-monitoring*. This construct reflects students’ ability to regulate their learning processes and is assessed through ten Likert-scale items from question ST309 (e.g., “I stop to think before I act”). Students rated their agreement on a five-point scale from strongly disagree (1) to strongly agree (5). This construct focuses on metacognitive strategies, such as planning, monitoring, and adjusting behaviors, to foster purposeful and autonomous learning. The other is *proactivity toward learning*. This construct captures students’ sustained effort and persistence in mathematics. Assessed via eight items from question ST293 (e.g., “I actively participated in group discussions during mathematics class”), students reported the frequency of these behaviors on a scale ranging from never or almost never (1) to all or almost all the time (5). Responses were aggregated into the Proactive Mathematics Learning Behavior index (MATHPERS; $\alpha = 0.82$), which quantifies adaptive engagement and goal-directed effort. These SRL constructs were chosen to reflect key aspects of self-regulation that are important for both academic success and lifelong learning. By emphasizing self-monitoring, proactive engagement, and adaptive strategies, these constructs align with PISA’s goals of preparing learners to face evolving global challenges.

In addition to SRL constructs, the study incorporates the *Economic, Social, and Cultural Status* (ESCS) index, a composite measure derived from parental education, occupation, income, and cultural resources. The ESCS index helps contextualize SRL behaviors within structural inequities, providing insights into how socioeconomic factors intersect with self-regulation strategies. This inclusion allows the study to distinguish the impact of self-regulation on achievement from the broader systemic influences of socioeconomic background.

Data Process and Analysis

The analysis used a comparative design to contrast SRL behaviors and mathematics achievement between East Asian systems and the OECD average. Descriptive statistics (means, SDs) were computed to quantify cross-system differences. Quantile regression models were employed to examine how SRL behaviors predict achievement at varying proficiency levels, focusing on the 25th and 75th percentiles to capture variations in performance. This method is particularly useful for analyzing relationships across the distribution of achievement, providing insights into how SRL behaviors affect students at both high and low achievement levels.

Separate models were run for each East Asian economy to account for system-specific dynamics, with ESCS included to isolate the unique contributions of SRL. By focusing on distributional extremes, this approach illuminates whether SRL behaviors disproportionately benefit high- or low-achieving students, offering insights into potential equity implications.

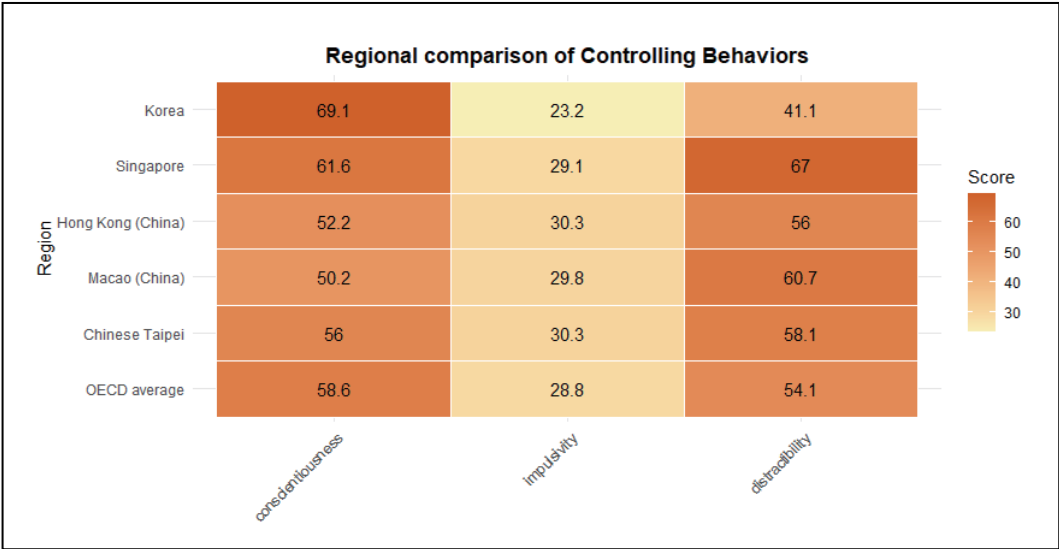
The integration of sampling weights (i.e., W_FSTUWT and W_FSTR1–80) and plausible value (i.e., PV1MATH–PV10MATH) methodologies ensured statistical precision, aligning with PISA’s rigorous reporting standards. These methods provide a robust framework for understanding how SRL behaviors contribute to mathematics achievement in diverse educational settings, and how these contributions may vary across different proficiency levels.

Results

Students’ Controlling of Learning and Self-Monitoring Behaviors

A comparative analysis of self-reported controlling behaviors across six East Asian education systems and the OECD average highlights significant regional differences in conscientiousness, impulsivity, and distractibility (see Figure 1). These variations are influenced by distinct cultural norms, educational practices, and societal expectations surrounding self-regulation and academic behaviors. The results offer valuable insights into how students’ self-regulation differs across these diverse educational systems.

Korea emerges as a clear outlier, exhibiting exceptionally high levels of conscientiousness, particularly in academic tasks. Notably, 89.6% of Korean students report a strong emphasis on producing error-free work, a figure substantially higher than the OECD average of 64.2%. This commitment to academic rigor is reflected in students’ self-reported behaviors; for example, 78.0% of Korean students are conscientious in verbal communication. However, there are notable inconsistencies in students’ self-monitoring practices, as only 58.9% of Korean students consistently proofread their homework. This discrepancy suggests that despite the strong emphasis on the outcome (error-free work), the process of reviewing and self-monitoring might be underemphasized in Korea’s education system. The intense focus on high-stakes testing and final assessments likely prioritizes performance results over procedural diligence, which may explain this divergence in conscientiousness and self-monitoring behaviors.



Note. Data are not available for Japan.

Figure 1. Regional comparison of controlling behaviors

Chinese Taipei demonstrates intermediate levels of conscientiousness, with 62.2% of students indicating a strong focus on producing error-free work. However, only 38.0% of students from Chinese Taipei engage in regular homework review, a figure well below the OECD average of 44.3%. This gap suggests a relative lack of emphasis on formative assessment practices in Chinese Taipei, which may hinder the development of consistent self-monitoring behaviors. The moderate level of conscientiousness observed in this region contrasts with the relatively high academic expectations placed on students, further indicating that conscientiousness in Chinese Taipei may be more focused on final outputs rather than the ongoing process of academic reflection and adjustment.

Singapore, on the other hand, excels in social self-regulation, with 70.9% of students demonstrating patience in waiting their turn to speak. However, the country's academic conscientiousness is comparatively weaker, with only 42.5% of students reporting that they regularly check their homework. This pattern suggests that while Singaporean students are highly attuned to social norms and group harmony, the emphasis on individual academic behaviors such as self-monitoring and review is less pronounced. This finding may reflect a broader societal tendency in Singapore to prioritize collaborative and social competencies over individual academic diligence, which is often a central feature of Western educational paradigms.

Similar patterns of academic conscientiousness are observed in Hong Kong (China) and Macao (China), where social self-regulation tends to be relatively strong, but academic self-monitoring behaviors are notably weaker. For instance, only 37.0% of students in Hong Kong (China) and 40.0% in Macao (China) report proofreading their homework, both figures below the OECD average of 44.3%. These findings are consistent with the collectivist cultural values prevalent in these regions, where social harmony and group cohesion are often prioritized over individual academic efforts. This cultural emphasis on cooperation and social regulation may account for the observed discrepancy between students' academic behaviors and their social self-regulation.

In terms of impulsivity, significant regional variation is also observed. Hong Kong (China) and Macao (China) report notably higher levels of impulsivity, with 36.9% and 35.6% of students, respectively, engaging in impulsive speaking behaviors. This tendency toward impulsivity may reflect cultural norms that prioritize quick responses and active participation in discussions. In contrast, Korea displays the lowest levels of impulsivity, with only 30.1% of students acknowledging impulsive speaking behaviors. This lower level of impulsivity in Korea aligns with the country's cultural emphasis on deliberate reflection and careful academic preparation, where students are encouraged to take time to think critically before responding.

Overall, these regional differences in self-regulation, conscientiousness, impulsivity, and distractibility provide valuable insights into how diverse cultural and educational contexts shape students' controlling behaviors in learning. By examining these behaviors across different East Asian economies, the study underscores the complex interaction between societal values, educational practices, and individual learning strategies, which collectively influence student outcomes in high-stakes academic environments.

Students' Proactive Mathematics Study Behaviors

The PISA 2022 assessment of proactive mathematics study behaviors reveals significant regional variations in SRL behaviors, shedding light on how students across different East Asian educational systems engage with mathematics outside of formal classroom settings. These differences in proactive study behaviors are influenced by cultural norms, teaching practices, and students' individual

learning strategies.

Among the regions assessed, Singapore demonstrates the most pronounced proactive engagement in mathematics learning. A substantial 78.7% of Singaporean students report actively paying attention during lessons, while 48.8% make meaningful connections between new content and prior knowledge—figures that exceed the OECD averages. This strong proactive engagement is further reflected in Singapore's high levels of effort and time management, with 75.8% of students putting consistent effort into assignments and 60.0% exhibiting effective time management skills. These behaviors contribute to Singapore's robust proactive mathematics study profile, as reflected in the Proactive Mathematics Study Behavior index (MATHPERS), where Singapore achieves the highest score ($M = 0.21$, $SD = 0.92$).

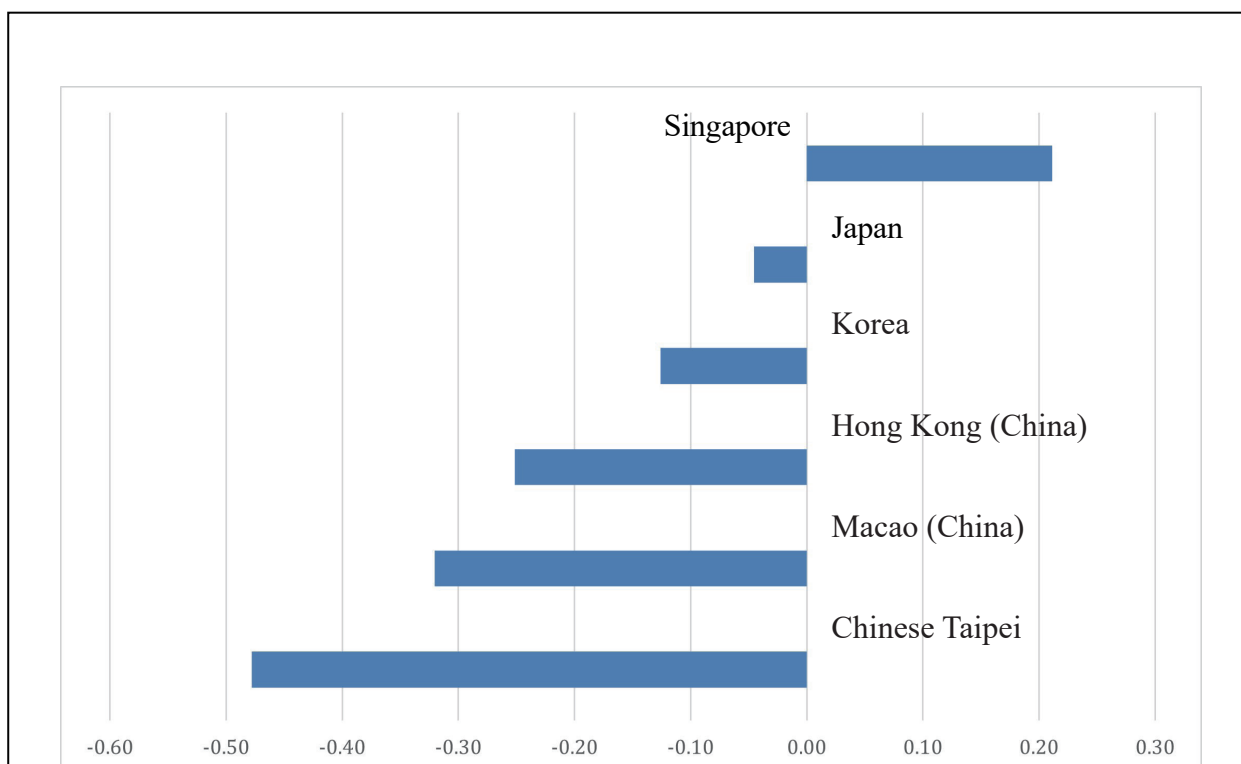
Japan also shows commendable levels of active participation, with 73.2% of students paying attention during class, and 39.2% engaging in group discussions. However, Japan (27.2%) lags behind Singapore in making connections between new and prior material. Despite a strong overall effort—75.7% of Japanese students invest effort in assignments—the country's time management skills are comparatively weaker, with only 43.1% of students demonstrating effective time management. This suggests a discrepancy between students' effort and their ability to manage study time effectively.

Korea exhibits moderate levels of proactive mathematics engagement, with 68.0% of students paying attention in class and 36.5% participating in group discussions. The country's performance in effort and time management also falls within the moderate range, with a significant portion of students reporting consistent effort, although time management skills are not as strongly developed as in Singapore. These findings suggest that while Korean students are engaged, there may be room for improvement in developing strategies to optimize study time and enhance learning outcomes.

In contrast, Hong Kong (China), Macao (China), and Chinese Taipei report relatively lower levels of proactive engagement in mathematics. Approximately one-third of students in these regions report making connections between new and prior knowledge, reflecting a more passive approach to learning compared to other East Asian economies. Additionally, both Hong Kong and Macao show lower levels of effort, with fewer students demonstrating consistent effort in completing assignments. These regions also exhibit less pronounced time management skills, which may contribute to their relatively weaker performance in the MATHPERS index. Chinese Taipei reports the lowest score in the MATHPERS index (-0.48), indicating comparatively weaker proactive learning behaviors.

Regarding motivation and attitude toward mathematics, Singapore stands out for its low levels of disengagement, with only 17.6% of students reporting a decline in interest during lessons, significantly lower than the OECD average of 30.2%. Hong Kong and Macao also report relatively low levels of disengagement, highlighting a relatively high degree of sustained interest in mathematics among students in these regions. In contrast, Korea and Chinese Taipei show slightly higher levels of disinterest, suggesting potential challenges in maintaining students' intrinsic motivation during lessons.

The regional differences observed in the proactive mathematics study behaviors are further illustrated by the MATHPERS index (see Figure 2), which provides a composite measure of proactive engagement. Singapore's high score reflects its students' strong proactive learning behaviors, while Chinese Taipei's low score highlights areas where further support may be needed to enhance students' self-regulation and study habits.



Note. The red line indicates the OECD average, set as 0.

Figure 2. Proactive learning index across East Asian regions

Relationship between SRL Behaviors and Mathematics Achievement

The relationship between SRL behaviors and mathematics achievement is notably strong, particularly for students at lower performance levels. In East Asia, conscientious SRL behaviors are positively correlated with higher mathematics achievement, while impulsivity and distractibility tend to have more pronounced negative effects compared to the OECD average.

In Korea, high levels of conscientiousness are closely linked to substantial performance gains. For instance, students who prioritize producing error-free work and engaging in clear verbal communication demonstrate significant academic advantages. At the 25th percentile, students with conscientious SRL behaviors score +22 points higher, and at the 75th percentile, they achieve a gain of +12 points. These results highlight the effectiveness of conscientious behaviors in enhancing mathematics achievement. However, the lower rate of homework checking (58.9%) suggests that certain educational and cultural factors may limit the full potential of these behaviors. This discrepancy can be attributed to Korea's intense academic pressure, which emphasizes exam performance over process-oriented learning and self-monitoring practices. As a result, although conscientious behaviors are valued, the lack of emphasis on formative assessment, such as regular homework review, may restrict the development of fully effective SRL practices.

In contrast, Singapore's proactive learning behaviors, such as active participation in lessons and effective time management, contribute to its strong position in mathematics achievement. Students who consistently invest effort in assignments achieve a gain of +37 points at the 25th percentile and

+30 points at the 75th percentile. These findings reflect Singapore's emphasis on a holistic educational approach that balances academic rigor with the cultivation of self-regulation skills. The proactive learning environment in Singapore fosters not only cognitive skills but also the development of behaviors essential for long-term academic success.

Impulsivity, a key SRL behavior that typically has negative consequences, is particularly detrimental to mathematics achievement in East Asia. In Korea, impulsive students experience a significant decline of -17 points at the 25th percentile, while in Singapore, the drop is -15 points. These findings suggest that impulsivity disproportionately affects lower-performing students in these regions, potentially exacerbated by the high-stakes nature of their educational systems. The immediate pressures and competitive nature of East Asian education may contribute to the heightened impact of impulsivity on academic performance, especially among students who are already struggling academically.

Distractibility also has a notable, though somewhat variable, impact across regions. In Korea, the negative effect of distractibility is more pronounced, leading to a performance decrease of -10 points, which is greater than the OECD average decline of -5 points. Conversely, in Hong Kong (China) and Singapore, the impact of distractibility is less severe, with some positive changes observed among higher-performing students. These regional differences may be attributed to differences in classroom environments and teaching methods. Korea's highly structured and rigid educational system may exacerbate the detrimental effects of distractibility, while other regions with more flexible teaching styles may allow students to manage their attention and focus in ways that buffer the negative impact of distractibility.

Proactive learning behaviors, such as active participation and time management, are strongly associated with improved mathematics achievement, particularly at higher percentiles. In Korea, students who exhibit these behaviors show an impressive gain of +41 points at the 25th percentile and +33 points at the 75th percentile, highlighting the critical role of engagement in enhancing academic performance. Japan demonstrates more modest improvements in mathematics achievement linked to SRL behaviors, while Hong Kong (China) and Macao (China) show smaller but still meaningful changes in performance. These findings underscore the importance of fostering proactive learning behaviors across different educational systems.

Time management and effort are closely linked to performance, with Korea once again leading the way, showing a remarkable gain of +62 points. Hong Kong (China) and Chinese Taipei also exhibit moderate gains of +51 and +42 points, respectively. These results emphasize the critical role of SRL behaviors, such as effective time management and consistent effort, in boosting mathematics achievement.

The regional differences in SRL behaviors and their impact on mathematics performance can be attributed to distinct cultural and educational contexts. For example, Korea's emphasis on academic rigor and conscientiousness reflects Confucian values that prioritize diligence and self-discipline, which are seen as essential for academic success. However, the lower rate of homework checking suggests that the high-pressure academic environment may discourage students from engaging in process-oriented behaviors such as regular review and self-monitoring. In contrast, Singapore's holistic approach to education fosters proactive learning behaviors and mitigates the negative effects of impulsivity and distractibility. This approach reflects Singapore's broader educational philosophy, which aims to develop well-rounded individuals capable of thriving in an increasingly complex and dynamic global environment.

In summary, the relationship between SRL behaviors and mathematics achievement is complex and multifaceted, with significant regional variation. Conscientious behaviors, such as effort and time management, are strongly correlated with higher achievement, particularly in Korea and Singapore. Conversely, impulsivity and distractibility have more severe negative effects in East Asia than the OECD average, underscoring the importance of mitigating these behaviors to support student success. By understanding how SRL behaviors interact with regional educational practices and cultural contexts, policymakers and educators can design interventions that foster more effective learning strategies, ultimately improving mathematics outcomes across diverse educational systems.

Conclusions and Discussions

Regional Variations in SRL and Academic Achievement

This investigation elucidates the intricate relationship between self-regulated learning (SRL) behaviors and mathematics achievement within East Asian educational contexts. Despite observable disparities in SRL engagement across regions, students in East Asia consistently surpass their OECD counterparts in international assessments such as PISA and TIMSS (OECD, 2019; Mullis et al., 2020). This paradox—where regions exhibit relatively lower SRL in specific domains yet achieve high academic outcomes—suggests that factors beyond SRL significantly influence performance. Institutional frameworks, cultural values, and pedagogical rigor collectively underpin these outcomes (Zhao, 2021).

For instance, countries like Korea and Singapore demonstrate elevated levels of conscientiousness and proactive SRL behaviors, whereas Hong Kong (China) and Macao (China) manifest less emphasis on domains such as homework checking and self-monitoring (Tan & Ng, 2021). Nonetheless, these regions maintain exemplary academic results, challenging traditional assumptions that prioritize SRL as the central determinant of success. Instead, the findings underscore the importance of contextual factors, including systemic structures and cultural norms, which shape educational achievement (Hattie, 2009).

The observed regional disparities prompt critical questions regarding the broader educational ecosystem. In Korea and Singapore, success appears to be reinforced by institutional attributes—rigorous examination regimes and high academic standards—that foster discipline and effort (Kim & Park, 2020). Korea's exam-centric system, coupled with a strong emphasis on academic integrity, cultivates a culture of deliberate practice that compensates for less developed SRL behaviors (Lee, 2018). Similarly, Singapore's structured curriculum and clear instructional guidance sustain engagement despite comparatively lower levels of autonomous time management (Ng, 2021). These insights highlight the necessity of viewing SRL as a component within a complex ecosystem where systemic supports are pivotal to achievement (Zimmerman, 2013).

Furthermore, divergences among East Asian regions—particularly Singapore, Korea, and Chinese Taipei—highlight how varying instructional philosophies interact with cultural values to influence SRL. Singapore's emphasis on active participation and strategic time management directly correlates with academic success (Tan & Ng, 2021). Conversely, Korean students tend to exhibit higher engagement levels, whereas Taiwanese students demonstrate lower proactive behaviors, especially in homework checking—underscoring how cultural and pedagogical paradigms shape SRL profiles (Zhao, 2021).

The Role of Cultural and Systemic Factors

Cultural values rooted in Confucian philosophy serve as foundational influences on SRL behaviors in East Asia. These traditions valorize effort, discipline, respect for authority, and collective responsibility, shaping students' approaches to learning (Lee & Shute, 2010). Such cultural paradigms foster motivation driven by moral and familial obligations rather than purely intrinsic interests, which, although often characterized as extrinsic, can effectively catalyze achievement (Markus & Kitayama, 1991).

For example, the Confucian emphasis on hierarchy may constrain autonomous learning and metacognitive reflection, as students are socialized to defer to teachers and follow directives without critical questioning (Hofstede, 2011). This deference can limit internal feedback loops essential for SRL components such as self-monitoring and strategic adjustment. Nevertheless, these cultural norms also promote perseverance, task persistence, and conscientiousness—traits aligning with certain SRL dimensions like goal-oriented behavior and sustained engagement (Hattie, 2009). Thus, cultural values shape SRL expressions in context-dependent ways rather than opposing its core principles.

Systemic factors further influence SRL development. In East Asia, parental involvement extends beyond support to structured monitoring, tutoring, and collaboration with schools (Ho & Willms, 1996). Such external scaffolding can compensate for limited intrinsic SRL capacities, effectively outsourcing self-regulation mechanisms within ecological frameworks (Fan & Chen, 2001). Peer interactions—marked by competitive and collaborative dynamics—further modulate SRL behaviors. Highly competitive systems reinforce individual performance goals, while the school environment—characterized by structured schedules, frequent assessments, and after-school programs—creates a conducive milieu for SRL practices, often supplemented by private tutoring centers (“juku” or “hagwons”) that provide additional regulated learning contexts (Rohlen & LeTendre, 1998).

In contrast, Western educational models—such as those prevalent in the U.S. and Europe—prioritize autonomy, intrinsic motivation, and exploratory inquiry (Deci & Ryan, 2000). These paradigms conceptualize SRL as a key twenty-first-century skill, cultivated through inquiry-based approaches and personalized assessments. However, this construct assumes a baseline of learner agency and self-efficacy, which may not be uniformly present across student populations. Notably, East Asian systems leverage systemic support structures to achieve high achievement levels despite uneven SRL engagement, reinforcing the notion that cultural and institutional scaffolds are integral to student success (Zhao, 2021). This contrast underscores the importance of culturally sensitive conceptualizations of SRL that acknowledge the interplay of individual agency and systemic affordances.

Policy Implications and Recommendations

The findings underscore the necessity of designing educational policies that are attuned to regional cultural and systemic contexts. Rather than adopting universal SRL models, policymakers should recognize SRL as a dynamically embedded construct, mediated by cultural norms and institutional frameworks (Zimmerman, 2013). Enhancing SRL should involve context-sensitive strategies that complement existing pedagogical and systemic strengths.

In Korea, where students exhibit high conscientiousness but limited self-monitoring, policy initiatives could focus on developing metacognitive skills and self-assessment literacy through reflective practices integrated into classroom routines (Kim & Park, 2020). Teacher professional development programs should foster dialogic pedagogy and formative assessment techniques that model SRL behaviors in authentic contexts (Hattie, 2009).

Chinese Taipei, despite its rigorous academic environment, could benefit from bolstering executive functions like time management and strategic planning. Implementing targeted workshops and leveraging digital tools—such as learning dashboards and real-time feedback applications—can cultivate self-regulation habits that complement traditional academic rigor (Huang & Chen, 2020; Azevedo & Cromley, 2004). Incorporating collaborative activities with leadership roles can further promote peer-regulated learning and accountability (Johnson & Johnson, 1999).

Singapore's relatively advanced SRL profile invites policies aimed at deepening metacognitive reflection and interdisciplinary applications. Curricular innovations encouraging students to justify reasoning, critique peer work, and engage iterative revisions foster higher-order SRL skills (Flavell, 1979). Embedding real-world problems and inquiry-based tasks can deepen engagement, facilitating transferability of SRL competencies beyond mathematics (Jonassen, 2000).

Crucially, these strategies should not solely target immediate academic outcomes but also aim to develop lifelong learning capabilities—an imperative in a rapidly evolving global economy (Zimmerman, 2002). Incorporating SRL training within national curricula supports both current achievement and future-readiness, aligning pedagogical objectives with societal needs for adaptable and autonomous learners (Dweck, 2006; OECD, 2018).

Future Research Directions

The complex interplay of SRL and achievement invites multiple avenues for further inquiry. One promising direction involves examining how systemic and ecological factors—such as family support, peer culture, and school climate—interact with SRL behaviors across socioeconomic strata and cultures (Epstein, 2018). Longitudinal research can elucidate developmental trajectories and the long-term impact of early SRL interventions on academic and occupational outcomes (Schunk & Zimmerman, 2007).

Cross-cultural comparative studies are essential for refining culturally grounded models of SRL. While Western frameworks have dominated the discourse, understanding how non-Western paradigms shape SRL development can inform more inclusive and effective pedagogies (Alexander, 2001). Comparative analyses might reveal alternative strategies for fostering agency within culturally embedded norms, providing insights adaptable to diverse contexts (Deci & Ryan, 2000).

The integration of technological tools in SRL promotion presents significant opportunities. Adaptive learning environments, digital feedback systems, and AI-driven tutoring have the potential to personalize regulation support, but empirical validation of their effectiveness across learner populations remains limited (Azevedo & Cromley, 2004; Wong et al., 2019). Further research should assess how these innovations impact self-regulation in diverse educational settings.

Lastly, addressing socioeconomic disparities in SRL development is critical. In resource-constrained environments, innovative, low-cost interventions—such as community-based programs, culturally responsive pedagogy, and accessible technology—could mitigate achievement gaps and promote inclusivity (Gorski, 2013; Sirin, 2005). Understanding how to foster SRL in marginalized populations will be vital for creating equitable educational futures.

In sum, advancing our understanding of SRL necessitates an integrative approach—one that considers cultural, systemic, and socioeconomic dimensions—aimed at informing contextually relevant policies and pedagogies. Such efforts will better support learners in cultivating the self-regulatory skills essential for academic success and lifelong adaptability.

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CREATING MATHEMATICS CLASSROOMS WHERE ALL STUDENTS CAN SUCCEED AND ENJOY LEARNING

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In a typical classroom, students possess diverse abilities and knowledge levels. Ignoring this reality and assuming uniformity among students often leads to ineffective teaching. Unfortunately, many mathematics classes adopt this approach by setting learning goals tailored to average-performing students, thereby neglecting the needs of both advanced and struggling learners. This presentation explores strategies for designing inclusive mathematics classrooms that accommodate diverse intellectual abilities. Key topics include redefining learning goals, leveraging tasks with multiple entry and exit points, and implementing practices that encourage differentiated learning. Practical examples and video demonstrations from one of my 2018 third-grade addition lessons will illustrate how these strategies can be applied effectively. The discussion will also emphasize the interconnected operation of these elements to foster a classroom environment where all students can achieve success and experience joy in learning mathematics.

Keywords: Tasks with a variety of entry and exit points, Practices, Teachers' beliefs

Introduction

The majority of classrooms comprise approximately twenty students. Mathematics teachers are required to conduct instruction for these students. These students are not computers with identical capabilities, but rather possess diverse abilities and knowledge, as well as differing attitudes toward learning. Instruction can be successful only when *it faithfully reflects these differences*. Conversely, if teachers ignore these differences for the sake of convenience, such instruction cannot be considered effective. Teachers who aim to embrace and address these differences must adopt instructional practices that diverge significantly from traditional methods. Integrating practices that acknowledge these differences into conventional teaching methods, while preserving their essence, does not constitute instruction that truly acknowledges the individual needs of students.

Conducting instruction that recognizes these differences entails creating a learning environment in which all students actively engage. There are various aspects to consider in order to realize such instruction. Not only is it difficult to address all these aspects in this brief presentation, but it is even more challenging to account for the organic interactions among them. Although this discussion may be insufficient, it will focus on some of the teacher beliefs related to instruction in which all students actively engage, tasks with various entry and exit points (TVEE), and instructional practices.

Teachers' Beliefs

Learning Objectives

NCTM (2014) and Smith and Stein (2011) emphasize that teachers must establish explicit and well-defined learning objectives when planning instruction. While the majority of educators would ostensibly concur with this perspective, the practical implementation of such objectives becomes considerably complex when one considers the heterogeneity of students within a classroom - students who possess divergent cognitive abilities, prior knowledge, and dispositions toward learning. The

question thus arises: For whom are these learning objectives intended? Are they designed for high - achieving students, those with lower achievement, students of average ability, or for the entire class? Is it reasonable to expect that all students will attain a singular learning outcome, simply because the teacher has articulated it and structured instruction accordingly? Our collective pedagogical experience suggests that such an expectation is unrealistic.

Recognizing this, how then should teachers - who seek to authentically address the intellectual diversity present within their classrooms - formulate learning objectives? Kim and colleagues (Jung & Kim, 2013; Kim, Colen, & Colen, 2013; Kim & Lee, 2012) contend that given learners' varying intellectual capacities, even when exposed to identical instructional experiences, the knowledge and understanding that each learner constructs will necessarily differ. Accordingly, they propose that the learning objective should be for each student to meaningfully reconstruct or extend their current understanding relative to their starting point. In practice, students inevitably construct idiosyncratic understandings, even within the same instructional context. Thus, regardless of how precisely a teacher delineates learning objectives and orchestrates instruction, the resultant cognitive constructions will be inherently diverse, and it is improbable that the teacher's articulated objective will be uniformly realized by all learners.

Human beings, irrespective of age or status as student or adult, possess unique cognitive schemas; consequently, even when engaging in analogous learning experiences, individuals construct distinct interpretations and meanings from them. It follows, then, that learning objectives must be formulated on an acknowledgment of this cognitive diversity. The fundamental purpose of student engagement in instruction is to facilitate a transformation in their current conceptual framework - what may be termed the growth and development of cognitive structures. To this end, each student must actively endeavor to make sense of the mathematical content addressed in instruction. Teachers, therefore, should design and implement tasks with multiple entry points, thereby enabling students to engage with mathematical ideas at varying levels of readiness and sophistication.

A further consideration in relation to learning objectives is the recognition that the learning objectives should not be conceived as isolated or discrete items of knowledge, but rather as complex and interconnected. Students should be afforded opportunities to interrogate and internalize conceptualizations and understandings constructed by themselves and their peers, thereby expanding their own cognitive repertoires beyond their initial constructions. Through such rich learning experiences, students' cognitive structures will evolve into robust, interconnected networks comprising well - understood concepts, developing understandings, knowledge yet to be assimilated, and even misconceptions or erroneous notions. In this way, the schema that emerges is characterized by increased adaptability and resilience (Skemp, 1987).

Students' Intellectual Abilities

Liljedahl (2020), after observing a variety of mathematics lessons, points out that a majority of students in mathematics classrooms "cease thinking" during instruction. This observation is also noted by Stevenson (1994) and Stigler and Hiebert (2009). Furthermore, in Korea, analyses of mathematics instruction have shown that most teacher questions follow the IRE (Initiation-Response-Evaluation) pattern or the funneling questioning pattern (Cho & Shin, 2010; Kang, Jang, & Jung, 2011). In classrooms where such questioning predominates, students are not the agents of their own thinking, but rather are simply following the teacher's line of thought. Boaler (2004) demonstrates that, even with the same task, it is possible to pose questions that promote thinking rather than relying on traditional textbook questions. Teachers who pose questions that require students to think independently hold a strong belief in their students' capacity for reasoning. Teachers who lack this

belief may consider their own explanations, followed by students' replication of those explanations, as evidence of student thinking. However, in classrooms where such instructional practices prevail, the majority of students are, in fact, not engaging in genuine thinking (Liljedahl, 2020).

Are elementary school students capable of thinking? More specifically, are students who have just entered elementary school capable of thinking? Furthermore, can students with lower academic achievement think? According to Piaget, children around the age of entering elementary school are forming the concept of number conservation. The formation of this concept demonstrates that they are capable of reasoning from their empirical experiences (Kamii, 2003). Kamii showed that children can construct the concept of conservation through interactions with non-conservers, even without adult intervention. In other words, students at the time of entering elementary school possess the intellectual capacity to construct knowledge.

Factors that influence the development of students' intellectual abilities include maturation and heredity, active experience, social interaction, and equilibration (Wadsworth, 1995). Since each individual's cognitive structure is determined by the complex interplay of these factors, no two individuals can have identical cognitive structures at any given moment. This is equally true for students within the same classroom. The initial cognitive state of each student sitting in a classroom is different. This statement also applies to first-grade students at the very beginning of their schooling. Even on the first day of first grade, before receiving any formal education, students possess different intellectual abilities and background knowledge. Students have the intellectual capacity to construct knowledge, and these capacities differ from one another. Conducting mathematics lessons that acknowledge and respect this diversity constitutes inclusive instruction.

Tasks with a Variety of Entry and Exit Points

Example of a task with a variety of entry and exit points

It is incontrovertible that students with varying intellectual abilities coexist within a single classroom. In order to facilitate learning for each student at their respective level, what kind of tasks should teachers prepare? Tasks designed for a specific group of students - such as those at an average, advanced, or lower level - are, by their nature, ill-suited for classrooms characterized by such diversity. However, a task that *allows each student to approach it in different ways and produce a range of outcomes* is well suited for a classroom composed of students with varying intellectual abilities. Such tasks are referred to as Tasks with a Variety of Entry and Exit Points (TVEE) (Kim, 2020a, 2023a; NCTM, 2014; Yeh, Ellis, & Hurtado, 2017). Lessons utilizing these tasks can be found in the works of Burns and colleagues (Burns, 2001, 2003, 2010) as well as Kim (2020b, 2021a, 2021b, 2023b).

A well-known example of TVEE is the "Today's Number" routine (HCPS Mathematics, 2025; Kim, 2020b, 2023b; Ministry of Education, 2024; Ronfeldt, 2003). The basic form of this task is to create expressions that result in a given target number. For example:

"Create an expression that equals 100."

From this basic format, an almost infinite variety of modified versions of the task can be devised, and teachers can easily create such variations. By adjusting the magnitude of the target number, the task can be adapted to suit each grade level and period. For example, at the beginning of first grade, it might be presented as "Create an addition equation that equals 10.", and by the end of the first semester, Today's Number might be 60 with three numbers and one addition and subtraction. Or for fourth graders, teachers can do Today's Number with the number 5 using fraction addition. This

kind of TVEE task can be made and used for any grade and time.

Let us examine whether this Today's Number task, "Create an expression that equals 10.", qualifies as a TVEE. Because there is no restriction to the sum of two numbers in this task, some students may use a trial-and-error strategy. For instance, a student might try $1+4$ to get 5, realize that this is not 10, add 4 again ($1+4+4$) to get 9, see that this is still not 10, then add 3 ($1+4+4+3$) to get 12, and since this exceeds 10, subtract 2 ($1+4+4+3-2$). It should not be assumed that first-grade students will simply add 1 to 9 after obtaining 9, because they are still in the process of learning and may not automatically recall that $9+1=10$. Of course, even among first graders, there may be students who remember and use $9+1=10$. While it is possible to use trial and error even when restricted to the sum of two numbers, by imposing no restrictions, students are able to approach the task based on their own natural reasoning. Approaching a given task with natural reasoning in order to solve it forms the basis of problem solving (Polya, 1980). In other words, this task allows for multiple entry points. Because the entry points differ, the mathematical knowledge each student engages with also differs. In this respect, the task also has multiple exit points. The following figures show the results obtained by Kim (2000) from third-grade students during the first mathematics lesson of the first semester.

3.7(4) 100 만들기
(243)

$10+90=100$	$250-50=100$	$10 \times 10=100$
$20+80=100$	$200-100=100$	$20 \times 5=100$
$30+70=100$	$250-150=100$	$50 \times 2=100$
$40+60=100$	$300-200=100$	$1 \times 100=100$
$50+50=100$	$350-250=100$	$2 \times 50=100$
$15+85=100$	$155-55=100$	
$25+75=100$	$255-155=100$	
$35+65=100$	$355-255=100$	
$45+55=100$	$455-355=100$	
$55+45=100$	$555-455=100$	
$50+80-30=100$	$100+10+80=100$	
$40+70-10=100$	$20+10+70=100$	
$30+90-20=100$	$30+10+60=100$	
$20+90+10=100$	$40+10+50=100$	

Figure 1. Soyeon's approaches

3.7.1

$50+50=100$	$100+10=100$	$20+80=100$	$70+30=100$
$100/40=100$	$200-100=100$	$101-1=100$	
$250-150=100$	$130-30=100$	$1000-900=100$	
$98+2=100$	$91+9=100$	$97+3=100$	$99+1=100$
$92+8=100$	$900+800=100$	$700-600=100$	
$0+100=100$	$1 \times 100=100$		

Figure 2. Yookyung's approaches

3.7.1 100 만들기

$50+50$	$60+40$	$70+30$	$80+20$	$90+10$
25×4	20×5	$1000 \div 10$	$10000 \div 100$	$100000 \div 1000$
100×1	$20000 \div 200$	$30000 \div 300$	$40000 \div 400$	
$99+1$	$98+2$	$97+3$	$96+4$	$1000-900$
$95+5$	$94+6$	$93+7$	$92+8$	$91+9$
$90+10$	$89+11$	$88+12$	$87+13$	$86+14$
$85+15$	$84+16$	$83+15$	$500 \div 5=100$	$99.5+0.5$

Figure 3. Sewon's approaches

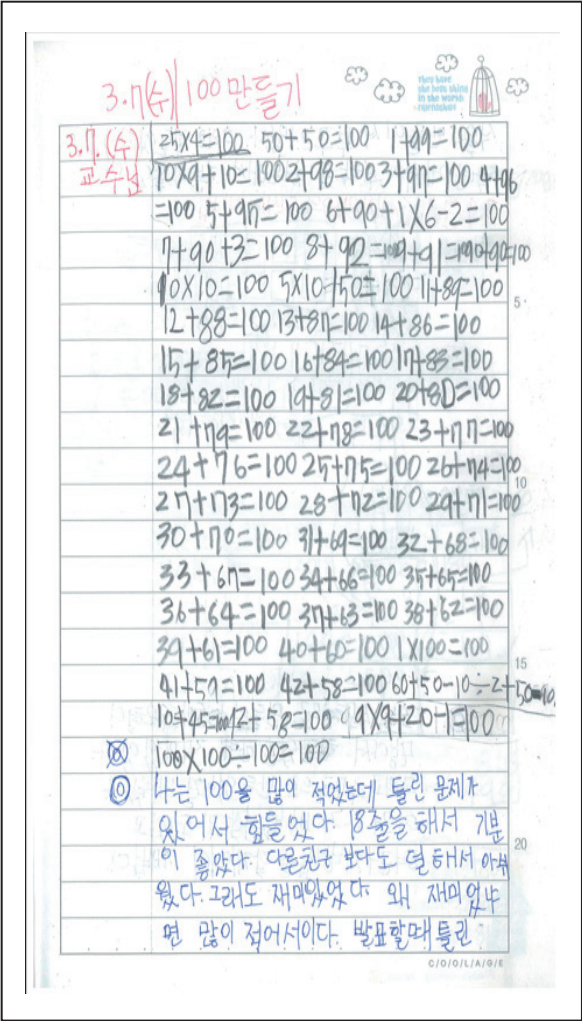


Figure 4. Jiwon’s approaches

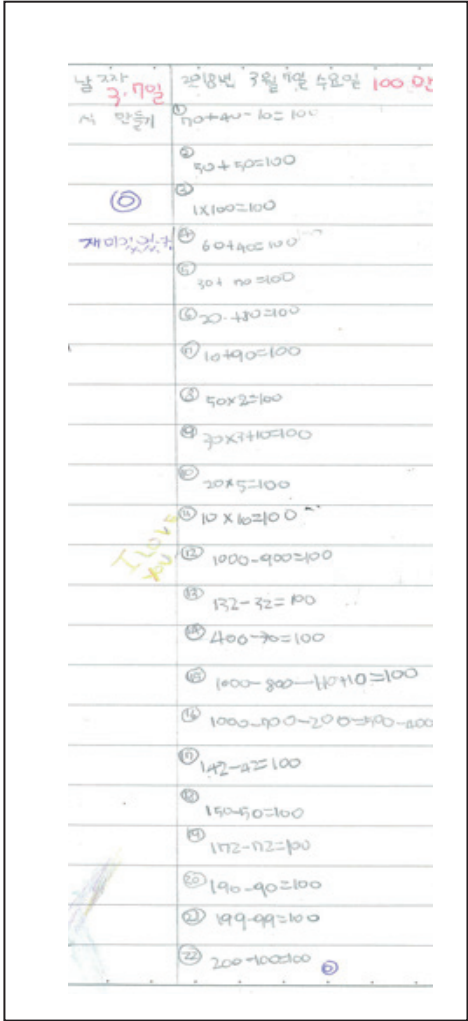


Figure 5. Seoyoung’s approaches

The fact that this was the first mathematics lesson of third grade means that these students had studied only two-digit addition and subtraction, as well as multiplication tables, during second grade. Nevertheless, the students demonstrated approaches such as addition of two or more numbers (for example, 30+10+60), subtraction of a three-digit number from a three-digit number (for example, 555-455), mixed addition and subtraction (for example, 20+90-10), mixed addition, subtraction, and multiplication (for example, 60+50-10÷2+50), decimal addition (99.5+0.5), multiplication and division (100×100÷100), and even the use of patterns in two-number addition division (as in Soyoon’s and Sewon’s approach of 1000÷10, 10000÷100, 20000÷200, 30000÷300, 40000÷400). It is also evident that students are not limited to a single operation but combine various operations, indicating that the knowledge within each student’s cognitive structure is interconnected.

These diverse approaches by each student represent their individual understanding. Thus, when using TVEE in instruction, what each student learns may differ.

Real Tasks in Lessons Using TVEEs

When teachers implement lessons where TVEEs are used and ideas generated at different levels by students are shared during class discussions, each student can engage with content they are inter-

ested in understanding. In other words, what students learn lies in the exchange of their individual understandings constructed while completing TVEE tasks. Since these understandings differ due to the knowledge embedded in their approaches being varied among students, during a series of lessons where these understandings are shared among peers, the knowledge encountered by students is highly diverse. For some students, this knowledge serves as review - content they already know; for others, it represents preview material - content they have never thought about; and for others still, it constitutes learnable content - knowledge they can comprehend and incorporate immediately as you can see from the figures above.

Through TVEEs, students simultaneously encounter these three types of knowledge: knowledge for review, knowledge for previewing future learning material, and knowledge for immediate learning. This means that within each student's cognitive structure exists highly interconnected knowledge, forming complex structures unique to each individual. The more important point is that because these experiences involve constructing their own understanding independently through such tasks over time, their capabilities develop further.

Kim, Colen, and Colen (2013) stated that the most important aspect of mathematics learning is creating new constructions based on one's prior constructions. In other words, true learning tasks are not the aforementioned TVEE itself but rather what students construct themselves during the tasks. The methods the students create become the material for study (Kim, 2014). This has two implications.

The first, as might be expected, is comprehending what peer learners have constructed. Just because students are provided with TVEE it's unlikely that each student will attempt all possible approaches. Therefore, as each student shares their approaches, their peers need to learn the thinking embedded in the strategies. Through the process of understanding ideas they had not considered themselves, students can expand their comprehension. The second finds patterns, similarities, and recognizes how each approach fits together based on what peer learners have constructed. When students share their diverse ideas with one another, they can discover commonalities among them. While inductive reasoning may be difficult with a small number of examples, having many examples makes inductive reasoning easier as you can see in the figures above. Furthermore, assuming that the shared ideas are valid, students can also engage in deductive reasoning. That is, students can use both inductive and deductive reasoning flexibly according to their needs during their knowledge construction processes to develop new understandings.

For such lessons to be possible, teachers must firmly believe that "even though they are elementary school students, they have the intellectual ability to construct knowledge." By recognizing that the knowledge each student constructs may differ, the tasks mentioned above can be effective when implemented in lessons.

Enrichment Tasks Derived from TVEE (Basic Tasks)

If we acknowledge that students have the ability to construct knowledge and that this ability varies among students, teachers need to prepare tasks for students who quickly complete the given basic tasks due to their advanced learning abilities - namely, enrichment tasks. Having students who finish basic tasks quickly assist those still working on them is an instructional practice rooted in the belief that all students must learn the same content. However, as previously mentioned, teachers should allow students to learn different content rather than the same content. During task performance, teachers must ensure that each student can work on tasks appropriate to their abilities. In other words, even highly capable students should be able to engage with tasks suited to their level during a single lesson, rather than being limited to basic tasks. For this reason, teachers need to pre-

pare enrichment tasks.

These enrichment tasks should be derived from the basic tasks. This ensures that during class discussions after task performance, students can connect their thoughts from both basic and advanced tasks, share them collectively, and enhance their understanding. In lessons that follow the model of Differentiated Instruction - where different tasks are provided for each level - it is challenging to create an environment where ideas constructed by students of varying abilities can be shared collectively.

Now, let us explore several methods for designing enrichment tasks. First, the simplest form of an enrichment task is “Solving Problems in Multiple Ways.” Every math problem - even closed-ended ones (Latterell, 2004) - can be solved in various ways. Of course, since we are considering “TVEEs,” these are inherently accessible through multiple approaches. Nevertheless, while students are solving problems, teachers should provide prompts such as “Try solving it in more ways; there might be other methods,” especially at the beginning of the semester for students who stop after solving it one or two ways. Through such teacher prompts, high-achieving students are likely to make efforts to utilize the full range of their existing knowledge during task performance, and in doing so, they will naturally attempt unique and original methods.

If high-achieving students restrict themselves to employing only familiar strategies with which they are already proficient, they are unable to cultivate what is commonly referred to as problem-solving competence. The critical transformation of a given task into an authentic learning opportunity occurs at the moment when, prompted in this way, students attempt to engage with the problem using approaches that are unfamiliar to them. Accordingly, at the outset of the academic term, teachers should consistently prompt students with directives such as “solve the problem using multiple methods,” so that, as a result, this practice of employing diverse solution strategies becomes embedded as a normative aspect of classroom culture. In other words, not only must the task itself possess the characteristic of being solvable in multiple ways, but the teacher must also continually emphasize this through questioning during instruction. Therefore, solving problems in multiple ways should become a general learning culture in the mathematics classroom.

Another enrichment task is “creating one’s own problems.” Students who finish the basic task quickly can modify the conditions of the given problem to create new problems. While working on the basic task, students naturally attempt to modify the given conditions to create more challenging problems for themselves (Kim, 2023). For example, when students are given the previously introduced task, “Today’s Number: Create an expression that equals 20,” after finding various expressions that equal 20, one student voluntarily asked, “Can I make an expression that equals 100?” When the teacher allowed it, the student proceeded to find an expression that equals 100. In this way, a self-imposed, challenging task motivates the problem-poser to persist and enjoy the challenge, even if they make mistakes along the way. Furthermore, they are willing to share their thought processes. When students modify the conditions of the given basic task to create new problems, teachers should permit them to select mathematical content that may go beyond the standards for their grade. This is because the fundamental purpose of enrichment tasks is to provide challenges for students who are capable of engaging with content that exceeds the achievement standards.

Third is comparing solution processes and strategies. When several students complete the basic task quickly, the teacher can have these students gather in a designated area of the classroom to share their approaches, check results, and modify their own solutions or attempt their peers’ strategies. This peer sharing not only promotes reflection and metacognition but also allows students still working on the task to focus on their own solutions without distraction. Peer review and collabo-

rative comparison are recognized as effective strategies for developing critical thinking and problem-solving skills in mathematics classrooms.

Fourth is finding the total sum of data collected by groups. If the basic task is conducted in groups, in other words, students can calculate the sum of all values obtained by each group. While this type of task slightly diverges from the typical intent of enrichment tasks—since it provides all students with the opportunity to engage in a challenging activity—it still serves as an enrichment task by offering every student a chance to tackle a more complex problem. Although some students may experience frustration during this process, the collective effort and challenge foster a sense of enjoyment and community within the learning environment. An example of such a task that can be given to first-grade students is as follows: Each group is asked to determine the number of family members living together in their respective households, and then to sum the number of family members obtained by each group (Kim, 2023b). In other words, the task is to find the total number of family members represented by all students in the class. The sum will likely approach 100. Since the students have not yet learned the standard addition algorithm, they will be confronted with the task of summing the family numbers from five or six groups, and thus will be compelled to devise their own creative strategies rather than relying on familiar procedures. It is likely that, in such situations, the strategies employed by high-achieving students may be somewhat more systematic. The majority of students, when presented with five or six numbers, will add them sequentially from the beginning, one by one. However, high-achieving students may decompose numbers to form multiples of ten, or may first sum the tens and units separately to obtain partial sums, and then combine these partial sums to reach the final total. Kamii (1994) noted that such computational processes are analogous to the strategies employed by humanity prior to the development of standard algorithms, and that it is beneficial for students to experience these strategies, as doing so fosters cognitive adaptability (Kim, 2005).

As illustrated, there are various methods for developing enrichment tasks, which is very easy. It should be emphasized once again that such enrichment tasks are intended for students who complete the basic task quickly; thus, teachers must immediately provide these students with enrichment tasks as they arise. With the exception of the fourth enrichment task mentioned above, teachers should note that most enrichment tasks are not to be provided additionally to high-achieving students only after all students have completed the basic task.

Some Characteristics of TVEE

The characteristics of TVEE, as intermittently mentioned in the previous section, can be summarized as follows. First, the diversity of entry points allows students with lower academic ability to approach the task to the best of their ability at their own level. Therefore, these students also have the opportunity to share their own approaches during the whole-class discussion time after task completion. The time these students spend striving to understand the content addressed throughout the lesson contributes to their academic achievement. Indeed, numerous studies (Kim & Kim, 2010; Jeong & Kim, 2013; Cho & Kim, 2011; Cha & Kim, 2021) have demonstrated improvements in the academic achievement of students with lower academic ability. Second, the diversity of exit points means that not all students are required to produce the same outcome; each student can generate their own product. In fact, this is a natural expectation resulting from the diversity of entry points. In other words, the fact that exit points are diverse distinguishes these tasks from those that can be solved in various ways but must ultimately yield the same result. The 2022 Revised Mathematics Curriculum (Ministry of Education, 2022) emphasizes that tasks provided to students should require a variety of outcomes. The curriculum's recommendation of tasks in which both the methods and the results can be diverse is a direct response to the recognition that students in a single class-

room possess a wide range of intellectual abilities.

Third, TVEE is conducive to the creation of enrichment tasks as mentioned above. During instruction, high-achieving students may complete the basic task quickly, and for these students, enrichment tasks are necessary. TVEE allows students to generate their own enrichment tasks during engagement, or allow the teacher to easily design such tasks. The fourth characteristic, which derives from the first, is that students can create their own problems. As this point has not been previously discussed, it will be examined in greater detail below.

To illustrate this, let us briefly examine the “One Gorilla” task, a task with TVEE developed by the author. This task is based on the book “ONE GORILLA” by Atsuko Morozumi (1990). The teacher reads the book from beginning to end with the students, and, while discussing the animals in the book and sharing personal experiences, captures the students’ interest. The teacher then presents the following TVEE basic task: “Please, choose three kinds of different animals among the tens and find the sum of them.” Students select various combinations of three animals for different reasons. Although all students are given the same task through the storybook “ONE GORILLA,” each student selects a different combination of three animals, so each can calculate the total number of animals independently, without reference to other students’ approaches. Furthermore, students may use a variety of aids, such as drawing, using fingers, or connecting cubes, while employing their chosen strategies. This task also naturally allows students to continue selecting different combinations of three animals. Moreover, high-achieving students may choose combinations involving more kinds of animals. Recognizing that students in the classroom possess a range of intellectual abilities, the teacher should prepare enrichment tasks for those who complete the basic task quickly; these latter tasks are intended for such students.

Tasks of this type are distinct from those described by Boaler and her colleagues (2021, 2024) and Liljedahl (2020), which are characterized by allowing multiple solution methods. Typically, a problem (task) consists of conditions (a situation) and a question, and most tasks with mathematical diversity presented by Boaler¹⁾ and Liljedahl have a fixed question. Thus, students solve the same question using various methods. However, TVEE tasks like “One Gorilla” are open in that students can select elements that meet the conditions within the given situation, resulting in each problem solver generating a different question. The previously mentioned “Make an equation that equals 100” is also a TVEE task in which each student selects a different question to solve. To clarify this point further, consider the following example (Kim & Lee, 2014, p. 160):

In Cheolsu’s class, there are 12 boys and 13 girls. Using this information, create a problem and solve it.

This task provides only information to students and instructs each to create a problem using the information in their own way and then solve it. In other words, while there is a question, each problem solver is to create a different question. The two tasks presented earlier do not explicitly state this point, but the tasks assigned to students are open-ended tasks requiring them to create and solve their own problems.

Thus, such tasks allow each student to have a problem of their own creation. Therefore, students have a deep understanding of their own problem and must independently devise a method for solving it. Because students solve problems using ideas they have conceived, they engage in the process with curiosity and genuine interest, fully investing themselves in determining whether their approach will work as intended. Other students will be grappling with the problems they have created, so there is no need to compare their strategies to those of others - at least while solving their own

1) Some of Boaler’s tasks (2021) allow students to selectively formulate questions.

problems. When teachers assign TVEE, students are given the same assignment, but in practice, each is working on a different problem. This makes such tasks particularly appropriate for classrooms composed of students with diverse intellectual abilities.

Instructional Practices that Enable All Students to Participate Joyfully

Teaching a class of students with diverse intellectual abilities means that the teacher is practicing a lesson in which all students generate the best new understanding possible from their current state of intellectual comprehension. Since students begin a lesson with different levels of understanding, it is not appropriate to expect them all to reach the same level of understanding by the end of the lesson. For this to be realized, teachers must ensure that every student is actively engaged in every moment of the lesson.

The prerequisite for such participation is that teachers must acknowledge that all students have the intellectual ability to construct knowledge on their own. Even first-grade students entering elementary school, as well as those with lower learning abilities, can construct knowledge in their own ways. Kamii (2004) even argues that knowledge construction is possible through communication among non-conservers. With this belief, as explained in detail in the previous section, let us examine the practices that teachers can use during and after implementing tasks, where students are given TVEE.

Practices for Facilitating Finding a Variety of Strategies

During task engagement, teachers must actively encourage students to attempt a range of strategies and methods. Some students may try only one or two approaches and, upon obtaining a result, consider the task complete. This tendency is especially prevalent before the classroom culture of exploring diverse strategies is firmly established. Therefore, at the beginning of the academic term, teachers should consistently provide TVEE, fostering a classroom culture where solving problems in various ways is the norm.

As this culture becomes established, students will increasingly attempt diverse approaches independently, even without explicit teacher prompting. To support this, teachers should frequently communicate early in the term that “students can have different ideas, and we should allow time for those who need more time to think.” For example, in everyday situations such as choosing a field trip destination, students naturally offer a variety of suggestions. Teachers can leverage these experiences by inviting students to share their diverse opinions, helping them recognize that having different perspectives is both normal and valuable - even in mathematics.

Because students are provided with TVEE, they are naturally inclined to try their own methods. To further promote this, teachers should, during task engagement, highlight unique strategies observed among students with prompts such as, “OO is using a method that no one else is trying,” or by specifically naming the strategy. When a student discovers many approaches during individual, pair, or group work, the teacher can say, “Wow! OO has found two whole pages of different methods!” Such comments encourage other students to explore more strategies themselves. Even if a student’s approach is systematic but not flexible, reviewing these methods during whole-class discussions helps all students experience cognitive flexibility.

Early in the term, teachers should also inform students during class discussions that, “Even if you do not raise your hand, I may call on you to share your thoughts. It is not only the students who raise their hands who will be invited to speak” (Kim, 2023b). This helps students understand that they should actively engage with the mathematical content being discussed. Merely sharing one’s

own thoughts without listening to peers does not constitute deep thinking (Chapin, O'Connor, & Anderson, 2013). If a student who has not raised their hand is called upon and hesitates or declines to answer, the teacher should avoid reprimanding them. Instead, use prompts such as, "Next time, please try to share your thoughts," "Shall we listen to a friend's response again?" or "Can anyone else share?" This approach ensures that all students can participate in class without feeling embarrassed, supporting an inclusive and reflective classroom environment.

Practices for Actively Engaging Students with Low Learning Ability

Even teachers accustomed to traditional instruction recognize the considerable differences in intellectual ability and prior knowledge among students within a classroom; however, because it is difficult to fully reflect these differences in instruction, there is a tendency to focus lessons on students of average ability (Kim et al., 2024). As a result, students with low learning ability are inevitably marginalized. Consequently, these students are unable to achieve understanding through regular instruction and are often regrouped for remedial instruction in after-school programs. Lee (2020) and Boaler (2024), among others, have reported that instruction utilizing open-ended tasks in such after-school programs contributes to improved academic achievement for these students.

Nevertheless, students with low learning ability are also the responsibility of the teacher during regular classroom instruction. Teachers must design lessons and employ practices that actively engage these students. When introducing a task, the teacher should not simply provide unilateral explanations or rely on the assistance of high-achieving students, but should intentionally call on students with low learning ability to participate in demonstrations. This approach encourages these students to participate actively in class. Such instructional practices help students recognize that all are protagonists in the classroom, and that learning is not the exclusive domain of high-achieving students. Furthermore, when preparing tasks prior to the lesson, the teacher should prepare handouts outlining the procedures students must follow, and provide these after demonstrations, display them via TV monitor, or write them on the blackboard during the demonstration. This enables all students, not only those with low learning ability, to refer to these aids while focusing on their own thinking during task completion.

During whole-class discussions following task completion, the teacher should continue to call on students with low learning ability. Engaging in classroom discourse based on their incomplete responses can be as challenging as "walking through a quagmire" (Chapin et al., 2013; Kim, 2023b). Nevertheless, teachers must persist in actively involving these students. By consistently calling on students with low learning ability, teachers foster their participation in class, which, in turn, allows these students to achieve success in their own way.

Practice for Facilitating Students who Do Not Wish to Participate in Whole-Class Discussion

During instruction, teachers inevitably encounter students who are reluctant to participate actively in whole-class discussions. However, if such students remain disengaged even by the end of the academic year, it is necessary for the teacher to critically examine the instructional practices being employed by him/herself. If these practices are appropriate, it is expected that, by the end of the year, even these students will, to varying degrees, have become more actively engaged in class discussions (Kim, 2020, 2021a, 2021b, 2023b). Just as teachers utilize specific practices to actively engage students with low learning ability, they must also maintain an intentional stance toward encouraging participation among reluctant students, and this disposition should be conveyed to students through the teacher's instructional practices.

Teachers should monitor the thought processes of these students as they engage with tasks and, during whole-class discussions, encourage students who have not raised their hands to share their approaches at appropriate times. Even when prompted, these students may initially respond by waving their hands dismissively or saying, “I didn’t raise my hand!” or “I don’t want to.” When students exhibit such attitudes, teachers should accept them as they are, but also communicate the importance of sharing during class by saying, for example, “I hope you will present next time.” It is crucial that teachers do not respond negatively to such reactions, as doing so may further discourage students from developing the courage to share their thinking in front of their peers.

From the perspective of these students, presenting their thoughts to the entire class requires considerable courage, unlike conversing with friends in everyday situations, and this courage may not be fostered by a single instance of teacher prompting. By regularly inviting shy or reluctant students to participate, and observing as some gradually begin to share, teachers can facilitate the active engagement of most such students in classroom discussions.

4.4 Providing Students Enough Time to Share Their Thoughts

If one acknowledges that students possess the intellectual capacity to construct knowledge independently and that there exist differences in these intellectual capacities, then teachers must ensure that all students are given “sufficient time to think” during task engagement or whole-class discussions. Given that students are provided with tasks characterized by TVEE, as described in the previous section, teachers can adjust the time allotted for task completion according to students’ progress. One of the features of TVEE is that it is not necessary for every student to complete the task at the same time. Teachers should encourage students to engage with the assigned task to the best of their ability. It is not feasible to wait until all students have completed the task before proceeding (Kim, 2025). As previously mentioned, teachers should call on a wide range of students during whole-class discussions, regardless of their academic ability.

At this point, it may not be easy for students to articulate their activities verbally. If a student encounters difficulty in explaining their thinking, the teacher should prompt with statements such as, “Take your time to organize your thoughts,” “Would you like to try explaining again a little later?” or “What is difficult for you to express?” Such prompts help students maintain composure and organize their thoughts for presentation. Alternatively, the teacher may suggest, “Come to the board and explain while writing,” thereby allowing the student to utilize auxiliary tools such as drawing, using equations, or manipulatives to support their explanation. Through such encouragement, students may develop the sense that “the teacher respects my thinking” (Ministry of Education, 2024; Boaler, 2022). The formation of this sense of being respected serves as a driving force for students’ autonomous participation in class.

When a student who wishes to share becomes stuck, other students may volunteer to express that student’s thoughts or offer their own ideas. In such cases, the teacher should not immediately grant them the floor. Instead, the teacher should say, “Right now, [student] is speaking. Let’s give [the student] a bit more time. After [the student] has finished, I will give everyone a chance to share their thoughts.” Such teacher responses are crucial for establishing a classroom culture in which students actively communicate and learn to respect one another’s ideas.

Practices for Facilitating All Volunteers, Not the Students Who Are Picked by Teachers During Their Working time

Smith and Stein (2011) recommend that during students’ engagement with a task, teachers observe the ideas students generate and then select those students who have produced ideas deemed worthy

of discussion to share their thinking. They argue that seeking volunteers to present is tantamount to relinquishing control of the discussion.

But is this truly the case? According to Kim (2020, 2021a, 2021b, 2023b), in classrooms where the teacher has established a culture from the beginning of the semester that respects students' thinking as it is, students are eager to share their own ideas during task engagement and, in whole-class discussions, actively wish to share the new ideas they construct based on the contributions of their peers. From the teacher's perspective, the understandings constructed by each student may vary in quality, and among them, there may be ideas relevant to the teacher's instructional objectives. However, if the teacher centers whole-class discussions only on such selected ideas, students whose ideas are not chosen become peripheral participants rather than active ones. The more negative consequence is that, as students strive in hopes that their ideas will be selected by the teacher, most of their contributions may not be considered valuable for discussion and are ultimately ignored. As these experiences accumulate, these students become marginalized in classroom discourse. Each student's idea is valuable in its own right, and both teachers and peers should cultivate a classroom culture that respects them as they are. For students with lower academic achievement, the ideas they construct represent the results of their best efforts from their own perspectives. When these are acknowledged by teachers and peers as they are, such students gradually become more active and autonomous constructors of knowledge.

Smith and Stein (2011) further express concern that allowing unfiltered ideas to be shared may lead whole-class discussions in unproductive directions or make it difficult for other students to understand. However, Kamii (1992) argues that providing opportunities to share ideas that are difficult for others to understand actually enhances the understanding of the student who produced them. Furthermore, Boaler (2024) and Kim (2023b) assert that, by providing opportunities to collaboratively resolve such challenging ideas with peers, students develop the resilience to overcome mistakes. The ultimate purpose of schooling is for students to learn new knowledge that they do not yet know, and it is natural for students to generate confused or incomplete ideas in this process. Teachers must actively address these ideas during instruction.

Focusing on the Exchanges of Ideas Students Invent

Because students possess the intellectual capacity to construct knowledge and their intellectual abilities differ, their understanding will inevitably vary even when engaging with TVEE. The range of these differences can be considerable, especially in classrooms with a wide spectrum of learning abilities. For instance, as seen in the "creating an expression equal to 100" task, while some students may use addition and subtraction, others may employ multiplication, division, decimals, or complex mixed operations - approaches that typical students might not consider. Likewise, as previously noted, even a student's approach involving the addition of two two-digit numbers should be valued and shared, as it is meaningful from that student's perspective.

Should the lesson be finished simply because many students have shared their ideas? The answer is no. In fact, the real lesson begins now. The teacher must facilitate communication that promotes metacognition regarding peers' constructed ideas. *Students need to develop an understanding of their peers' thinking.* Since elementary students have limited working memory, teachers should support this by recording students' ideas on the blackboard or having students do so themselves as they share.

In addition, for students to exchange their ideas with one another, a classroom culture of attentive listening must be well established as a basic norm. To foster a classroom culture of listening, students should speak loudly enough so that even the student seated farthest from them can hear.

When a student speaks, the teacher should check with students who are seated far away to see if they can hear the speaker's voice clearly. If they cannot, the teacher should ask the speaker to speak louder, or prompt the student who could not hear well to request that the speaker speak more loudly. Through repeated experiences of this process at the beginning of the semester, students should acquire the norm that speaking loudly is a basic rule when speaking in the classroom.

For fostering a classroom culture of attentive listening, there is also the practice of restating a peer's idea in one's own words, which serves the function of understanding the ideas of fellow learners. When a peer presents an idea, the teacher may ask the students, "Is there anyone who can restate what [student] said in their own words?" In order for learners to respond to this question, they must have listened carefully when their peer was speaking; without listening, it is impossible to respond. Communication involves not only expressing one's own thoughts in a way that others can understand, but also thoroughly understanding the thoughts of others. Therefore, teachers should frequently prompt students to restate the speaker's words in their own language.

If a student struggles to verbalize their thinking, even though it is their own, the teacher should ask questions such as, "Is there anyone who can further develop the contributor's thinking so far?" or "Can someone explain what could be done next after what the speaker has said?" This allows peers to collaboratively advance the original idea. The teacher should not intervene to resolve the impasse; since the task allows for multiple entry and exit points, it is likely that another student has had a similar thought and can assist. If no student can resolve the impasse, it is preferable to move on and assign the unresolved part as homework. Homework, in this context, is not rote repetition of what was explained in class, but an opportunity for students to independently resolve difficulties or strive to understand ideas that were not fully grasped during the lesson. The teacher should also involve the original contributor in this process, supporting his/her understanding as peers work through the impasse. Through these experiences, students come to see themselves as members of a learning community that constructs knowledge collaboratively, fostering a sense of belonging and mutual respect, as they recognize that each person influences others' learning.

In fact, for the teacher to facilitate this authentic lesson, it is necessary that ideas are recorded on the board, whether by the teacher or by the students, as they share their individual ideas. When students share, communication may occur solely verbally, or through the use of an overhead projector or TV monitor. However, with these methods of sharing, as soon as the next student begins to share, the previous student's content is replaced by the new student's content. This means that, in the students' memories as well, previous ideas fade as they focus on the new student's idea. Metacognition, that is, thinking about thinking, is important - especially thinking about the ideas of peers. For this reason, the ideas shared by students should remain on the blackboard in written form.

While a peer is sharing their idea and writing it on the blackboard, students who are seated at their desks should make an effort to understand this idea, and according to their needs. For example, if it is a very unique idea, or an idea they had considered attempting but did not, they focus on the ideas that interest them and record them in their notebooks. Some students write down all the ideas that have been shared on the blackboard. Of course, some other students, having been given TVEE, continue to develop and refine their own ideas.

The teacher must determine an appropriate timing at which to share ideas that students have generated while working on the basic task. Since students have been given TVEE, it is impossible to have all the various ideas constructed by students shared. In reality, even while students are sharing, they may continue to generate new ideas. When teaching with TVEE, it is, in fact, impossible for students not to be engaged in thinking throughout the entire lesson.

From the collection of ideas that fill the blackboard, a new lesson can emerge. Students can naturally explore similarities, differences, and possible transformations among the ideas before them - an instructional moment not possible with closed tasks that have a single correct answer. The teacher should facilitate this exploration by prompting students according to their interests and providing time for them to share their findings. In some cases, the shared outcomes become data for subsequent activities. For example, as mentioned above, if in a class of 24 students, six groups each determine the number of family members for their group and share these totals, the teacher can then pose the problem, “What is the total number of family members for our class?” The class can work together to solve this, or individual students can present their approaches. The teacher may also prompt students to identify commonalities among the strategies used by the six groups.

Closing Remarks

The reason students gather in the space of the classroom for lessons is to understand new knowledge that they did not previously know. However, because the intellectual abilities and prior knowledge of students in a single classroom differ, it is necessary to provide students with a task such as TVEE that can accommodate these differences. The various instructional practices required when teaching with TVEE, as introduced in this presentation, may be unfamiliar to teachers. As Kuhn (1970) discussed in relation to incommensurability, some teachers may not accept these practices at all, while others may attempt to add them to practices with which they are already familiar.

Nevertheless, these practices are optimized for instruction utilizing TVEE. When teaching with TVEE, *multiple forms of knowledge emerge within a single lesson*, and the learning objectives are realized through the growth and development of each student’s understanding. Furthermore, the agent of knowledge construction must be the student themselves. Although teachers recognize these differences, they are often reluctant to conduct lessons that fully reflect them. If instruction is conducted while denying or distorting this reality, it is not possible to expect positive outcomes, such as high academic achievement among students, the elimination of those who fall below basic academic standards, or the cultivation of positive attitudes toward mathematics learning.

It is hoped that an era will come in which the majority of teachers conduct lessons focused not on students at the middle level, but on all students.

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MATH IS FUN WITH BUNDLE-NUMBERS ON A BUNDLE-BUNDLE-BOARD

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Looking at four fingers held together two by two, we see four fingers, the essence. But, before school, children see what exists, bundles of twos in space, and two of them when counted in time. So, we may ask how mathematics may be taught to children if using their own two-dimensional bundle-numbers with units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by working with existence instead of listening to essence. Here we use the two core concepts of philosophical Existentialism holding that existence precedes essence. This will mean that counting precedes adding to bring outside totals inside by counting to be recounted and added. In this 'existence-math' approach, the mathematical essence is re-rooted as abstractions of outside existing examples instead of being defined as examples of inside abstractions. Hundred is a bundle-bundle, as four when recounted in 2s on a ten-by-ten BundleBundleBoard. The operations are icons created by counting processes and bundle-units make place values and carrying unneeded. Once counted, calculus and linearity are needed for next-to and on-top addition.

Keywords: mathematics education, arithmetic, early childhood, numeracy, existentialism

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as “ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.” Here the subgoal 4.6 wants to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”. However, different definitions of ‘numerate’ seem to exist. The English Oxford Dictionary defines it as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary defines it as “having the ability to understand and work with numbers.”

The difference in the definitions is interesting. The English uses the passive term ‘competent’ where the American uses the active term ‘work’. The word ‘competent’ is a predicate, a non-action word, I cannot ‘competent’ something, I can only be judged as competent by someone who is competent. In contrast, ‘work’ is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words ‘mathematics’ and ‘numbers.’ Again, mathematics is a non-action word, I cannot ‘mathematics’ or even ‘math’ something. In contrast, ‘number’ is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world as multiplicity in space and repetition in time. In contrast, mathematics does not do so, its concepts are institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions.

It thus seems that with the English definition of numeracy the assessment must be carried out by persons seen as experts on the predicates competent and mathematics. In contrast, the American definition of numeracy allows laymen to judge themselves the actions carried out when working with numbering and numbers. Also, in their common history, England once colonized America, so

we may wonder if the two different views are the views of a former colonizer and a former colonized.

To understand these differences and to enlighten and discuss the core of education formulated as ‘teach learners something’, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs and to discuss the ‘something’. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs and in the textbook-teacher-learner interaction. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs. Here, discussing may be a better word than enlightening since all three areas contain conflicting theories.

Grand Theory Looks at Mathematics Education

Within philosophy, a discussion between existentialism (Sartre 2007) and essentialism about the precedence of existence or essence has taken place since philosophy began in ancient Greece (Russell 1945). Here the ‘knowing’ sophists argued that to practice democracy we must tell nature from choice to avoid being patronized by choice masked as nature. In other words, we must be able to tell outside existence from its many chosen inside essences and especially the ones that have been institutionalized as the tradition and thus colonizes its outside existence (Habermas, 1981). Against this, the ‘better knowing’ philo-sophists argued that choice is nonexistent since everything physical is only imperfect examples of metaphysical essence as illustrated in Plato’s Cave allegory, and that essence is only accessible to philosophers educated at the Plato Academy.

This disagreement between sophists and philosophers about nature versus choice, and existence versus essence, runs through history. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. In the counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe’s line-organized office-preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the French Enlightenment republic’s post-structuralism inspired by Heidegger (1962) who argued that “P is Q” is a statement judging an outside existence, P, with an inside constructed essence-predicate that may be a preconceived prejudice, e.g., gossip (Gerede), and which therefore should be met with skepticism and be deconstructed. With ‘essence’ coming from Latin ‘esse’ meaning ‘being’, Heidegger gave four answers to the question “What is ‘is’?” pointing up, down, over, or nowhere: is an example of, is for example, is like, or is period. In mathematics, a function thus may be an example of a subset in a set-product where first-component identity implies second-component identity, or a name for calculation containing both specified and unspecified numbers, or a number-language sentence including a subject and a verb and a predicate as in a word-language sentence, or simply what it is, a stand-by calculation awaiting a number, e.g., ‘ $3 * x$ ’ in contrast to ‘ $3 * 5$ ’.

Psychology (Skemp, 1971) has a controversy within constructivism where Vygotsky (1986) sees education as adapting to the institutionalized essence-regime by building ladders down from it to the learners’ proximal learning zones. Piaget (1969) replaces this top-down view with a bottom-up view inspired by American Grounded Theory (Glaser et al, 1967) allowing categories to emerge

from concrete experiences. Siding with Piaget, Ausubel says that “The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly” (Ausubel, 1968, p. vi). Also siding with Piaget, Luhmann (1995) sees the brain as a closed self-referring psychic system inside a world of complexity from which it learns by itself reducing the complexity presented to it by another closed self-referring social system called education with its own reductions that it cannot export. Education thus means disturbing the students’ systems with outside complexity or hiding complexity if instead wanting uneducated students not able to reduce this, e.g., hiding that numbers occur differently in time and space. This conflict has sociological consequences. Presented top-down from above as examples of inside abstractions, concepts become hard to learn, which forces many learners to stop learning what is meaningless to them and to accept patronization by those who accept such meaninglessness. In contrast, bottom-up concepts grounded from below in the outside world are natural to learn for children through meeting and acting on the concrete examples that exemplify and root the concepts; and for teenagers since knowing the subject in a sentence creates automatic gossip-like learning.

Within Sociology, reviving the ancient Greek sophist skepticism towards patronization masking choice as nature, the Enlightenment created two democracies, one in North America still having its first republic, and one in France now having its fifth republic. In Germany, Weber (1930) was the first to theorize the increasing social goal-oriented rationalization that disenchants the world and create an ‘iron cage’ if carried to wide. Later, wanting to establish a third German Enlightenment democracy based upon communicative action and the convincing force of the better argument, Habermas (1981) accepts the role of science in rationalizing society, but warns against its instrumentalism spreading from the system to the life world to colonize it.

North American showed skepticism towards rationalist philosophy by developing American Pragmatism (Menand, 1997) leading to Symbolic Interactionism (Blumer, 1998) and Grounded Theory (Glaser et al, 1967).

Here, Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now” (p. 16).

Living together involves deciding upon which tasks to do individually and which to do collectively by creating rational institutions, “in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79).” However, Bauman warns against a possible ‘goal displacement’ where “The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).”

An example of a goal displacement may be institutionalizing a knowledge-regime (Foucault 1995) saying ‘The goal of mathematics education is to master mathematics’. By this self-reference, such a goal is meaningless. To master math cannot be the goal of math education, but it can be a means to reach the real end goal, to master Many. Which of course should be reached by different means if a goal displacement has changed institutionalized mathematics from a means to a goal itself colonizing any further road to mastery of Many and numeracy.

The debate on patronization beginning in ancient Greece between the philosophers and the sophists is still with us today between socialist top-down critical theorists and skeptical bottom-up existentialist theorists that are inspired by the French post-structuralist thinker Foucault saying

It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

So, seeing mathematics education as an institution (Freudenthal, 1973), a Weberian viewpoint would ask if the set-concept is a rationalization of Many gone too far thus leaving Many disenchanting and leaving the learners in an iron cage. And a Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. So, as an institution, mathematics is a means, hence the word ‘mathematics’ should go from goal descriptions. If not, there is a risk that mathematics education will practice ‘the banality of evil’ (Arendt, 1963). So, sociology would recommend replacing the truth regime saying ‘the goal of teaching math is to learn mastery of math’ by the real end goal, mastery of Many, e.g., by uncovering and developing the existing mastery of Many children create through adaptation to Many before school.

In short, it may be time to show skepticism towards institutionalized mathematics and use difference research (Tarp, 2018) to look for differences that may make a difference.

We therefore ask the Cinderella question: If mastery of Many is the end goal, essence-math may be a means for some but not for all as indicated by the more than fifty years since the first International Congress on Mathematical Education was held in 1969, but are there different means also, e.g., an existence-math, that will make the prince dance? And if so, what is its content and what kind of mathematics does the prince learn during the dance?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that $1+1=2$. In contrast, a different existence-math occurs when asking a 3-year-old child “How many years next time?”. The answer is four with four fingers shown. But seeing the four fingers together two by two, the child reacts: “That is not four, that is two 2s”.

This observation shows that numbers occur both as bundle-numbers in space, 2s, and counting-numbers in time, two, that need to be together since asking, “What is $1+1$?” leads to asking back “One what + one what?”. And asking, “What is $1s + 1s$?” leads to asking back “How many $1s$ + how many $1s$?”. And asking, “What is $1 + 1s$ ” leads to asking back “What??”

So now we will look for a different existence-math curriculum built on the bundle-numbers with units as 2 3s and 4 5s that children create before school when adapting to nature’s multiplicity in space and repetition in time? To gather data about the possible content of an existence-math curriculum we now use sociological imagination to visit a classroom in a not so far decolonized future.

Grade one Class one in a Decolonized Future

The teacher: Welcome children, I am your math teacher, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So ...

Showing a V-sign a child says: Mister teacher, here is one $1s$ in space, and here is also one $1s$. Now we count them in time to see how many $1s$ we have by saying ‘one, two’. So, we have two $1s$. But only until we add them as a bundle. Then we have one $2s$, so $1s$ plus $1s$ become $2s$, but one plus one is still one when we count it, and not two as you say. And together with this neighbor V-sign the total is one $2s$ plus two $1s$ which is one $4s$, and not three $3s$ as you will say.

And, if I show you three 1s two times then I have shown you six 1s. Counting-numbers as two and three can be multiplied, but they cannot be added without units since $2\text{ 1s} + 1\text{ 2s} = 2 * 1\text{s} + 1 * 2\text{s} = 1 * 4\text{s}$. So, we see that counting-numbers in time as 1 and 2 are different from bundle-numbers in space as 1s and 2s.

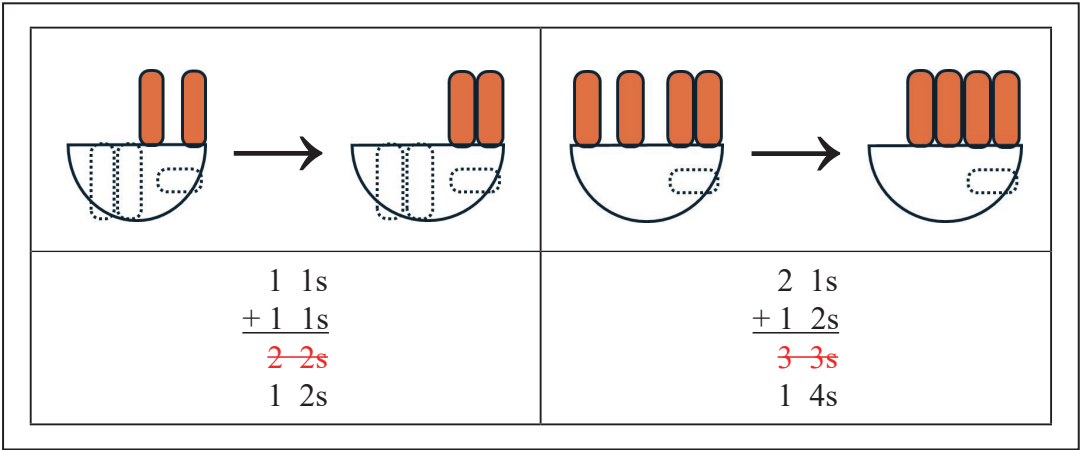


Figure 1. A closed and open V-sign show that $1+1 = 1$; and that $1\text{s}+2\text{s} = 4\text{s}$.

You would too if you ask a 3-year-old “How many years next time?” The child will say ‘four’ and show four fingers. Then, if you show the child the four fingers held together two by two, the child will protest: “That is not four, that is ...”

Do you know what the child will say? You can ask other adults or students in primary or secondary school. They will all say ‘four’ because they all see the essence that they have been adapting to ever since they started in school where they meet the culture that they are loyal to. And, they don’t mind having their brain colonized by whatever essence the culture has chosen to follow, which my uncle who is a philosopher calls an essence regime. But, before meeting the school’s culture and its essence, children adapt to the nature where they grow up and here they meet and adapt to existence.

So, when we come to school, we don’t want to know what $1+1$ as you would like to teach us. We at once ask “one what plus one what?” And $1+1$ certainly cannot be 2 as the collapsing V-sign shows. Instead, we show you two 2s on one hand and one 3s on the other hand and ask you how to add them. Then you say “We cannot add if the units are different, but don’t worry for in math there are no units. And what you show me is four and three that add to seven.” Then we must ask you what unit you use. If you use 7 as a unit, we show you 0B 4 and 0B 3 7s that add to 0B 7 7s or 1B 0 7s. And if you use ten as the unit, we show you 0B 4 and 0B 3 tens that add to 0B 7 tens. My uncle says that, by its nature, multiplicity is multiplicative so there will always be units. And if the school uses ten as unit without telling us, it practices ten-centrism that shows disrespect to local units.

But there are many other units than tens. Looking into my right hand I see two fingers to the left and three fingers to the right. Then I bend one left finger and two right fingers. Now, 1 of 2 of the left are bent, but only 1 of 3 of the bent are left. So, you cannot just disrespect the units.

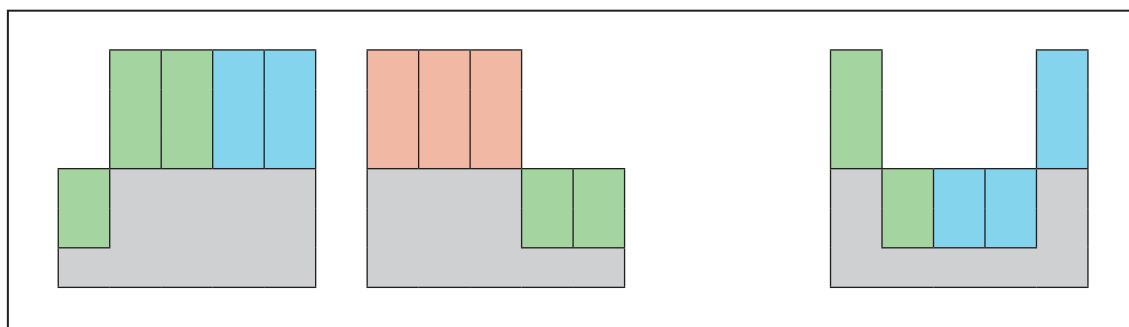


Figure 2. Two 2s in the left hand and one 3s in the right hand. And 1 left and 2 right bent.

So, let us now add two 2s and one 3s with respect to the units. We can see with our fingers that we can add them next-to as 5s and get one bundle and two 5s, 1B 2 5s. Or we can add them on-top as 3s or as 2s. If we add two 2s and one 3s as 3s we get two bundles and one 3s, 2B 1 3s. And if we add two 2s and one 3s as 2s we get 3 bundles and one 2s, 3B 1 2s.

But since we bundle in 2s, then the two 2s is one bundle of 2s or one Bundle of Bundles, 1 BB. So, the three bundles now are one bundle of bundles and one bundle and one unbundled, 1BB 1 B 1 2s.

|| || ||| → ||||| || || ||| → ||| ||| | || || ||| → || || || | → ||•|| || |

We can also show it with snap cubes on a table or on a ten-by-ten BundleBundle Board, a BB-Board.

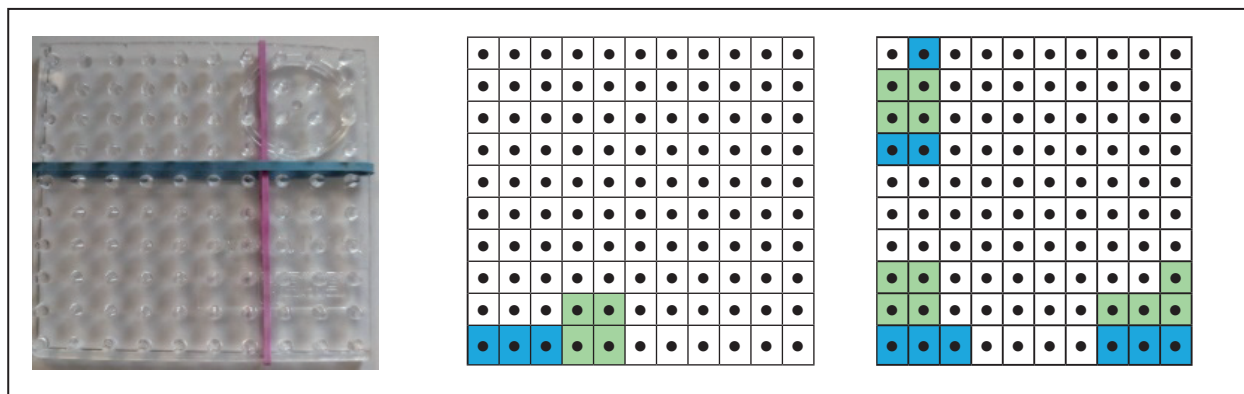


Figure 3. BundleBundle Boards with six 7s; and one 3s and two 2s added next-to as 5s or on-top

Therefore, mister teacher, please forget to add your line numbers without units. Instead, please help us add the bundle-numbers with units we bring to school, as two 3s and four 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on the BBBoard. If we add them next-to we add tiles, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says this is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school, instead of being colonized with your line numbers without units.

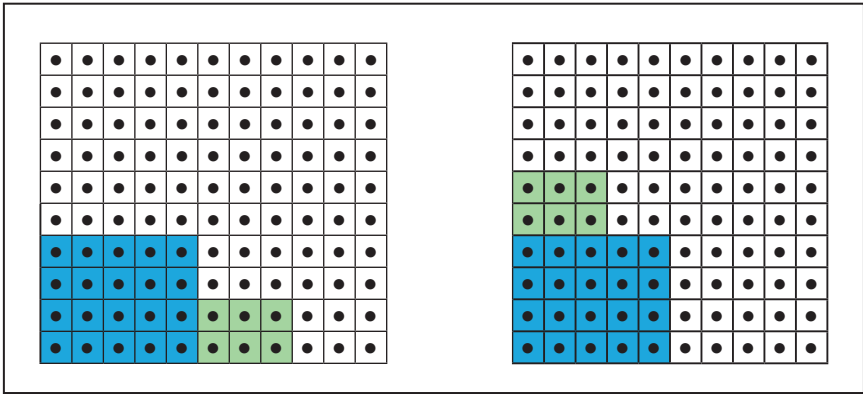


Figure 4. BundleBundle Boards with four 5s and three 2s added next-to as 8s or on-top as 5s

We know that you want to bundle in tens, and in ten-tens, and in ten-ten tens, but we like to bundle also in 2s, in 3s, in 4s, and in half-tens that we call half-bundles. You only write 47 and say ‘4ten7’ when you should write 4B 7 and say ‘4 bundles with ten per bundle and 7 unbundled ones’.

You say that 8 divided by 2 is 8 split in 2 parts. We agree that 1 8s split in 2 gives 2 4s, but we also say that 1 8s split in 2s gives 4 2s. So, to us, 8 divided by 2 mostly means 8 counted in 2s.



Figure 5. 1 8s split in 2 gives 2 4s, and 1 8s split in 2s gives 4 2s.

You cannot split 9 in 2 parts, but we can count 9 in 2s as 4 bundles and 1 unbundled that becomes a decimal, $9 = 4B\ 1\ 2s$. Or, with negative less-numbers in space or taken-away numbers in time, we get 5 bundles less 1 in space, and 5 bundles with 1 taken away in time, so here $9 = 5B - 1\ 2s$.



Figure 6. 1 9s split in 2s gives $9 = 4B\ 1\ 2s$, or $9 = 5B - 1\ 2s$

We know that you have not been taught this, and that the textbook doesn’t teach it. But don’t worry, we will teach you the way we learned to master Many in preschool.

Or better, instead of you colonizing our way let us find out together what math may grow from our bundle-numbers with units. My uncle uses the word existentialism when we let existence come before essence. So let us call our math for existence-math and your math for essence-math.

And, in existence-math, counting the totals must come before we later add them.

F01. Counting Totals in Bundles and Bundles-of-Bundles

Now, let us begin with bundle-counting the fingers on our hands. You only see the essence, five, but we see all the ways the five fingers may exist.

- Bundle-counted in 5s, a total of five fingers may exist as five 1s, $T = 0B\ 5\ 5s$, or as one bundle of 5s, $T = 1B\ 0\ 5s$.

|||||, or |||||

Bundle-counted in 5s, a total of ten fingers may exist as one bundle and five 1s, $T = 1B\ 5\ 5s$, or as two bundles, $T = 2B\ 0\ 5s$ or as two half-bundles tens, $T = 2\ \frac{1}{2}B\ 0\ tens$.

||||| |||||, or ||||| |||||

- Bundle-counted in 4s, a total of five fingers may exist as five 1s, $T = 0B\ 5\ 4s$, or as one bundle and one 1s, $T = 1B\ 1\ 4s$, or as two bundles less three, $T = 2B\ -3\ 4s$. Here, we use the words overload for $0B\ 5$, and underload for $2B\ -3$.

|||||, or |||||, or |||||

Bundle-counted in 4s, a total of ten fingers may exist as one bundle and six 1s, $T = 1B\ 6\ 4s$, or as two bundles and two ones, $T = 2B\ 2\ 4s$, or as three bundles less two ones, $T = 3B\ -2\ 4s$,

||||| |||||, or ||||| |||||, or ||||| |||||

- Bundle-counted in 3s, a total of five fingers may exist as five 1s, $T = 0B\ 5\ 3s$, or as one bundle and 2 unbundled, $T = 1B\ 2\ 3s$, or as two bundles less 1, $T = 2B\ -1\ 3s$.

|||||, or |||||, or |||||

Bundle-counted in 3s, a total of ten fingers may exist as one bundle and seven 1s, $T = 1B\ 7\ 3s$, or as two bundles and four 1s, $T = 2B\ 4\ 3s$, or as three bundles and one 1s, $T = 3B\ 1\ 3s$, or as four bundles less two 1s, $T = 4B\ -2\ 3s$, or as one bundle of bundles, and no bundles, and one 1s, $T = 1BB\ 0B\ 1$, since three bundles with 3 per bundle is one bundle of 3-bundles, or one BundleBundle of 3s, $1BB\ 3s$.

||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||

Here, power shows how many 1s the is in one 3-BundleBundle, $3^2 = 9$. And, the logarithm shows how many times a 3-Bundle has been bundled, $\log_3(9) = 2$.

- Bundle-counted in 2s, a total of five fingers may exist as five 1s, $T = 0B\ 5\ 2s$, or as one bundle and 3 unbundled, $T = 1B\ 3\ 2s$, or as two bundles and 1 unbundled, $T = 2B\ 1\ 2s$, or as three bundles less 1, $T = 3B\ -1\ 2s$, or as one bundle of bundles, and no bundles, and one 1s, $T = 1BB\ 0B\ 1$, since two bundles with 2 per bundle is one bundle of 2-bundles, or one BundleBundle of 2s, $1BB\ 2s$.

|||||, or |||||, or |||||, or |||||, or |||||

Bundle-counted in 2s, a total of ten fingers may exist as one bundle and eight 1s, $T = 1B\ 8\ 2s$, or as two bundles and six 1s, $T = 2B\ 6\ 2s$, or as three bundles and four 1s, $T = 3B\ 4\ 2s$, or as four bundles and two 1s, $T = 4B\ 2\ 2s$, or as five bundles and no 1s, $T = 5B\ 0\ 2s$, or as six bundles less two 1s, $T = 6B\ -2\ 2s$,

||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||, or ||||| |||||

Here, five 2s may also be bundled in 2s. First as 1 BundleBundle 3Bundles 0 2s, $T = 1BB\ 3B\ 0\ 2s$, then as 2 BundleBundle 1Bundles 0 2s, $T = 2BB\ 1B\ 0\ 2s$, and then as 1BundleBundleBundle 0Bun-

dleBundle 1Bundle 0 2s, T = 1BBB 0 BB 1B 0 2s.

|| || || || || , $\text{||}\bullet\text{|| || || ||}$, $\text{||}\bullet\text{|| ||}\bullet\text{|| ||}$, $\boxed{\text{||}\bullet\text{|| ||}\bullet\text{||}}\text{||}$

Again, power shows how many 1s there are in a 2-BB, $2^2 = 4$, and in a 2-BBB, $2^3 = 8$. And, the logarithm shows how many times a 2-Bundle has been bundled, $\log_2(4) = 2$, and $\log_2(8) = 3$.

• Bundle-counted in tens, a total of five fingers may exist as five 1s, $T = 0B\ 5\ \text{tens}$, or as one half-bundle, $T = \frac{1}{2}B\ 0\ \text{tens}$. And, a total of ten fingers will exist as one bundle and no unbundled, $T = 1B\ 0\ \text{tens}$, or as two half-bundles, $T = 2\frac{1}{2}B\ 0\ \text{tens}$. And, a total of twelve fingers (we include the two arms as fingers) will exist as one bundle and two unbundled, $T = 1B\ 2\ \text{tens}$, or as two half-bundles and two unbundled, $T = 2\frac{1}{2}B\ 2\ \text{tens}$.

Did you know that eleven and twelve come from the Vikings that counted ‘one left’ and ‘two left’? And that twenty comes from the Viking word for two tens, ‘two ti’.

You use the words ten, hundred, and thousand for B, BB, BBB, but you don’t have a special word for BBBB. You just call it ten thousand, but in China they call it one Wan, maybe because an army of hundred hundreds will win.

If we count in tens, forty-seven is 4B 7, and 547 is 5BB 4B 7.

$T = 4\text{tens} \ \& \ 7 = 4B\ 7\ \text{tens} = 47$
 $T = 5\ \text{hundreds} \ \& \ 4\ \text{tens} \ 7 = 5BB\ 4B\ 7\ \text{tens} = 547$

So, 547 is not one number but three numberings. You use a place value system, but, with bundles as the units, we do not need that.

Here we have counted totals in space, but we also use bundles as the unit when we count totals in time. If we count our five fingers in 3s we cannot say ‘1, 2, 3’, and so on since 1 is not 1 3s. Instead, it is 0 bundle 1 3s, so we count: 0B 1, 0B 2, 0B 3 or 1B 0, 1B 1, 1B 2 or 2B-1 3s. Or we may count: 1B -2, 1B -1, 1B 0, 2B -2, 2B -1 3s

F02. Digits and Operations are Icons for Degrees of Many and how we Act with them

With sticks we see that five 1s may be bundled as one 5s that may be rearranged as one 5-icon with five sticks. The other digits may also be seen as icons with the number of sticks they represent, where zero is a looking glass finding nothing. We don’t need an icon for ten since here the total is 1B 0 if we count in tens.

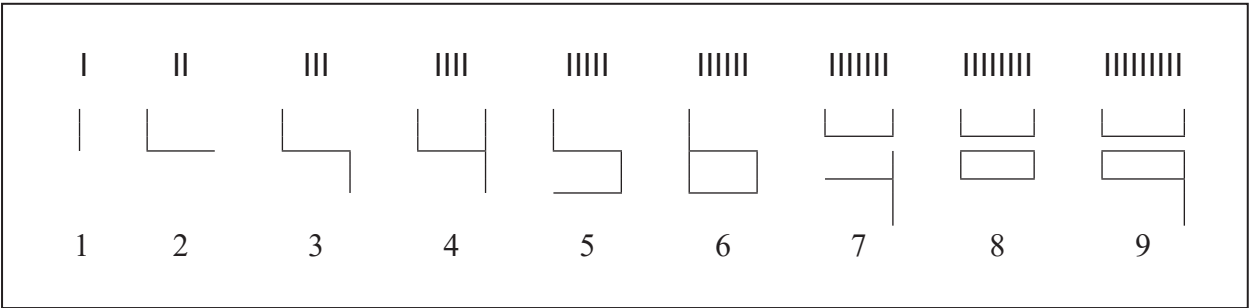


Figure 7. The bundles may be rearranged as icons with as many sticks or strokes as they represent

The calculations are icons also.

When counting eight fingers in 2s, division is a ‘push-away’ icon for a broom so that $8/2$ means ‘from 8 push-away 2s’. And the calculator can predict the result, $8/2 = 4$.

||||||| → ||||| || → |||| || || → || || || || → || || || ||

If we use snap cubes instead, after pushing-away bundles we can ‘push-back-to-stack’ the four 2s, which makes multiplication a ‘lift icon’ predicting the result of ‘4 times stacking 2s’, $4 \times 2 = 8$.

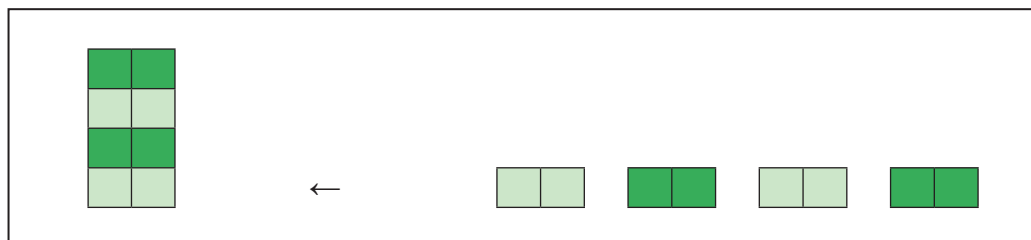


Figure 8. After counting 8 in 2s by pushing-away 2s, the four 2s are pushed-back-to-stack as 4×2

Since $8/2 = 4$, $8 = 4 \times 2$ may be rewritten as ‘ $8 = (8/2) \times 2$ ’ saying that 8 contains $8/2$ of 2s. With T and B for Total and Bundle we get a ‘recount formula’ that changes units.

$$T = (T/B) \times B \quad \text{or} \quad T = (T/B) * B.$$

My uncle calls it a very important formula for proportionality or linearity. Well, let’s look at minus.

How many fingers do I have here on my right hand? You say five, but we say ‘ALL’.

Now I bend my thumb down. How many fingers now? You say four, but we say ‘ALL minus 1’.

Now I bend my finger up. How many fingers now? You say five, but we say ‘ALL minus 1 plus 1’.

You learn it by bending the finger down and up several times while saying “ALL, is ALL minus one, plus one. ALL, is ALL minus one, plus one. We now write it down as ‘ $A = (A - 1) + 1$ ’.

Her we have shortened All to A, and we use a rope as an icon for pulling away, which is cancelled by a stroke when we pull back. So here we meet minus and plus as icons also.

Bending two or three fingers down and up we see that it also works with 2 fingers, and with 3 fingers, and with other bundles as well so we can now write down a ‘reunite formula’,

$$A = (A - B) + B.$$

F03. Recounting and Reuniting Solve Equations by Reversing Reversed Calculations

The recount formula, $T = (T/B) * B$, can solve equations as ‘ $u * 2 = 8$ ’ asking ‘How many 2s in 8?’ to which the answer, u , of course is found by recounting 8 in 2s as $8 = (8/2) * 2$, giving $u = 8/2$.

My uncle says this is correct since $8/2$ is defined as the number u that multiplied with 2 gives eight. And, on our hands we see that 8 contains the 4 2s that we pushed away when we counted 8 in 2s.

This we find by moving the known number across from the left side’s uniting process that can’t be done because of the unknown number, u , to the opposite right side’s splitting process.

Forward			Reversed			Forward			Reversed		
$u * 2$		=	8			$u + 2$		=	5		
$* 2$		$\uparrow \downarrow$	$/2$			$+ 2$		$\uparrow \downarrow$	$- 2$		
u		=	$8/2$			u		=	$5 - 2$		

Figure 9. Solving the equations $u * 2 = 8$ and $u + 2 = 5$: ‘To opposite side with opposite sign’

My uncle has warned us against a ‘same to both sides’ lever-method you might want to teach us. We don’t know what he is talking about, we just solve equations as ‘ $3 * u + 2 = 14$ ’ with a song

$3 * u + 2 = 14$ $(3 * u) + 2 = 14$ $3 * u = 14 - 2$ $u = (14 - 2)/3$ $u = 4$	Equations are the best we know; they’re solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, so only u itself will stay. We just keep on moving, we never give up. So feed us equations, we don’t want to stop.

Figure 10. Solving equations with the ‘to opposite side with opposite sign’ method

The reunite formula can solve equations as ‘ $5 = u + 2$ ’ where u is the unknown number. We just bend two fingers and say $5 = (5 - 2) + 2$, so $u = 5 - 2$. So, we just move plus 2 across as minus 2. My uncle says this is correct since $5 - 2$ is defined as the number u that added to two gives five.

Here I have ALL my fingers on the right hand. Now I bend two fingers down and get ALL-2. Which I now bent down also, so down I now have $2 + (ALL - 2)$ which is ALL. So, to get ALL down instead of up, I just added the difference, $ALL - 2$.

So, my number of down-fingers changed by adding the end-number minus the start-number, which is called the change of A, or ΔA if we use a triangle for change. So here, $\Delta A = \text{end} - \text{start} = A - 2$.

Now we have a change formula, $A = 2 + (A - 2) = 2 + \Delta A$, that works with all bundles:

$$A = B + \Delta A.$$

Now, if I know my start number and my change numbers, then I can find the end number by adding all the change-differences: Here, the start number is 3, and the change differences are 1, 2, 3, 2, 1 that add up to 9. So, the end number is, $3 + 9 = 12$.

By instead adding the change numbers one by one, we can find all the middle numbers

$$3, \text{ and } 3+1 = 4, \text{ and } 4+2 = 6, \text{ and } 6+3 = 9, \text{ and } 9+2 = 11, \text{ and } 11 + 1 = 12.$$

Or, if we know the middle numbers then we can find the change numbers as the differences

$12 - 11 = 1$, and $11 - 9 = 2$, and $9 - 6 = 3$, and $6 - 4 = 2$, and $4 - 3 = 1$,

Now we add all these differences and get $(12 - 11) + (11 - 9) + (9 - 6) + (6 - 4) + (4 - 3) = 12 - 3$.

T	$\Delta T = T2 - T1$	Sum of ΔT	Sum of $\Delta T = T6 - T1$
T1 = 3			
T2 = 4	$4 - 3 = 1$	1	
T3 = 6	$6 - 4 = 2$	$1 + 2 = 3$	
T4 = 9	$9 - 6 = 3$	$3 + 3 = 6$	
T5 = 11	$11 - 9 = 2$	$6 + 2 = 8$	
T6 = 12	$12 - 11 = 1$	$8 + 1 = 9 =$	12 - 3

Figure 11. Add Differences: $(T6 - T5) + (T5 - T4) + (T4 - T3) + (T3 - T2) + (T2 - T1) = T6 - T1$

Here we see that when adding many differences, all the middle terms disappear, so we only have to add one difference between the end-number and the start-number. This we will later use to solve difference equations in calculus.

F04. Recounting from Icons to Tens makes us Understand Times Tables better

Solving equations is an example of recounting from tens to icon-units. We will now look at the opposite, how to recount from icon-units to ten-units when asking, e.g., 6 7s is how many tens?

On a BundleBundleBoard we see $6 * 7$ as 6 7s that we can both see and feel. We see that 6 also occurs as $\frac{1}{2}$ B 1 tens and as 1B -4 tens, and that 7 also occurs as $\frac{1}{2}$ B 2 tens and as 1B -3 tens.

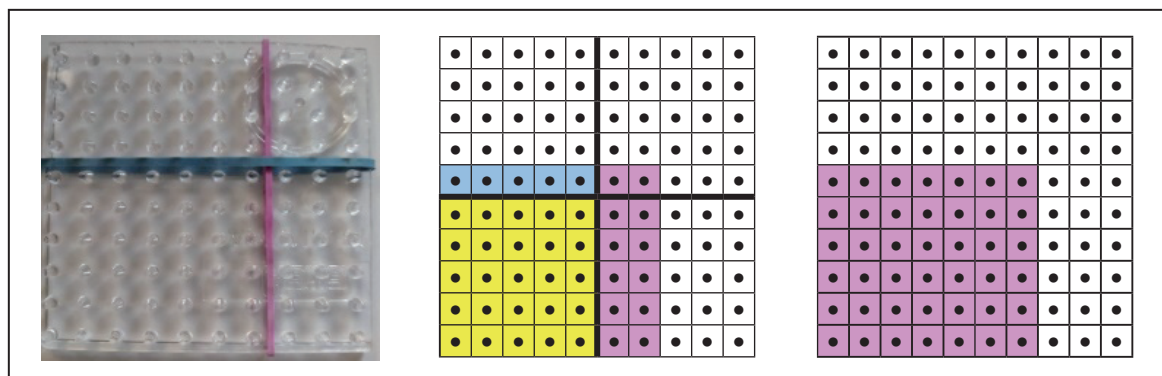


Figure 12. On our ten-by-ten BundleBundle Board we see $6*7$ as 6 7s; and 6 as $\frac{1}{2}$ B 1 tens and as 1B -4 tens, and 7 as $\frac{1}{2}$ B 2 tens and as 1B -3 tens.

- Method 01. Here, we let the fingers wander up the board touching the half-bundles while saying “ $\frac{1}{2}B$ 2 or 7, 1B 4 or 1ten4 or 14, 1B $\frac{1}{2}B$ 6 or 1B 11 or 2B 1 or 2ten1 or 21, 2B 8 or 2ten8 or 28, 2B $\frac{1}{2}B$ 1B or 3B $\frac{1}{2}B$ or 3B5 or 3ten 5 or 35, 3B 1B 2 or 4B 2 or 4ten2 or 42.”
- Method 02. We see that $7 = \frac{1}{2}B$ 2, so $6 * 7 = 6 * \frac{1}{2}B$ 2 = 3B 12 = 4B 2 = 4ten2 = 42
- Method 03. We see that $6 * 7$ is $(\frac{1}{2}B + 1) * (\frac{1}{2}B + 2)$, which is
 $= 2B \frac{1}{2}B$ (in the lower left corner) + $1 * \frac{1}{2}B$ (over) + $2 * \frac{1}{2}B$ (next-to) + 2 (in the right corner)
 $= 3B + 1B + 2 = 4B2 = 4ten2 = 42$.

So, while pointing or feeling we say “1, 2, 3, 4 Bundles and 2, so 4B2, or 4ten2.

With fingers, we show 6 as 1 finger up to the left, and 7 as 2 fingers up to the right. We don’t show the half-bundles. Then we add and multiply the up-fingers 1 and 2 into 3 and 2. Finally. We say “5, +3 is 8 $\frac{1}{2}$ bundles and 2, so 4B2, so 4ten2, so 42”.

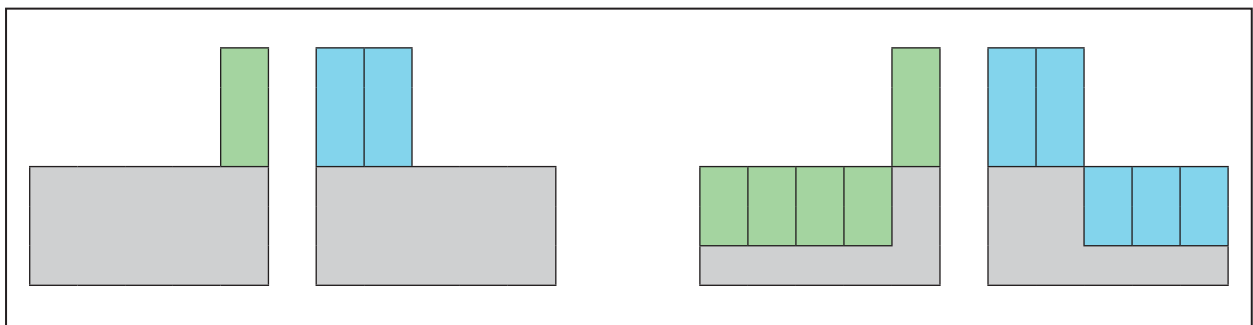


Figure 13. Six sevens or $6*7$ recounted in tens by adding and multiplying fingers.

- Method 04. We call this the ‘less-method’ since we pull-away the unneeded. My uncle calls it the Algebra method showing that minus times minus must be plus, and showing a FOIL method, First, Outside, Inside, Last.

$6 * 7 = (B - 4) * (B - 3) = 10B$ (all) – $3B$ (next-to) – $4B$ (on-top) + 4 3s (added since pulled away twice) = $(10 - 3 - 4)B$ 12 = $3B$ 12 = $4B$ 2 = $4ten$ 2 = 42. But $(10 - 3 - 4) = (5 - 3) + (5 - 4) = 2 + 1$ fingers where 7 is 3 fingers and 6 is 4 fingers down. So, with fingers, we show 6 as 1 finger up to the left, and 7 as 2 fingers up to the right. The up fingers we add as before and the down fingers we multiply. This gives $1+2 = 3$ and $3 * 4 = 12$. So, the result is $3B$ 12 or $4B$ 2 or $4ten$ 2 or 42.

TT1, the lower table from 2-5.

Use fingers, arms and toes.

The 2-table

$$2 * 2 = 2 \text{ } 2s = 4$$

$$3 * 2 = 3 \text{ } 2s = 6 = \frac{1}{2}B1$$

$$4 * 2 = 4 \text{ } 2s = 8 = \frac{1}{2}B3 = 2 * \frac{1}{2}B-1 = B-2 = 8$$

$$5 * 2 = 2 \text{ } 5s = 2 * \frac{1}{2}B = 1B0 = 10.$$

The 3-table

$$2 * 3 = 2 \text{ } 3s = 6 = \frac{1}{2}B1$$

$$3 * 3 = 3 \text{ } 3s = 9 = \frac{1}{2}B4 = 1B-1$$

$$4 * 3 = 2 * \frac{1}{2}B + 2 * \text{arm} = 1B2 = 12$$

(twelve = 'two left' in Viking language)

$$5 * 3 = 3 \text{ } 5s = 3 * \frac{1}{2}B = 1B5 = 15.$$

The 4-table

$$2 * 4 = 2 \text{ } 4s = 2 * \frac{1}{2}B1 = 1B-2 = 8$$

$$3 * 4 = 3 \text{ } 4s = 2 * \frac{1}{2}B + 2 * \text{arm} = 1B2 = 12$$

$$4 * 4 = 4 \text{ } 4s = 4 * \frac{1}{2}B-1 = 2B-4 = 1B6 = 16 =$$

fingers and toes less the outer ones

$$5 * 4 = 4 \text{ } 5s = 4 * \frac{1}{2}B = 2B0 = 20$$

(twenty = 'two tens' in Viking language)

•	•	•	•	•	•	•	•	•	•
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TT2, the middle table

Look at a BBBoard

The full 5-table

$$2 * 5 = 2 \text{ } 5s = 2 * \frac{1}{2}B0 = 1B0 = 10$$

$$3 * 5 = 3 \text{ } 5s = 3 * \frac{1}{2}B0 = 1B5 = 15$$

$$4 * 5 = 4 \text{ } 5s = 4 * \frac{1}{2}B0 = 2B0 = 20$$

$$5 * 5 = 5 \text{ } 5s = 5 * \frac{1}{2}B0 = 2B5 = 25$$

$$6 * 5 = 6 \text{ } 5s = 6 * \frac{1}{2}B0 = 3B0 = 30$$

$$7 * 5 = 7 \text{ } 5s = 7 * \frac{1}{2}B0 = 3B5 = 35$$

$$8 * 5 = 8 \text{ } 5s = 8 * \frac{1}{2}B0 = 4B0 = 40$$

$$9 * 5 = 9 \text{ } 5s = 9 * \frac{1}{2}B0 = 4B5 = 45$$

$$\text{ten} * 5 = 5 \text{ tens} = 5B0 = 50.$$

The 6-table

$$2 * 6 = 2 * \frac{1}{2}B1 = 1B2 = 12$$

$$3 * 6 = 3 * \frac{1}{2}B1 = 1B(5+3) = 1B8 = 18$$

$$4 * 6 = 4 * \frac{1}{2}B1 = 2B4 = 24$$

$$5 * 6 = 6 \text{ } 5s = 6 * \frac{1}{2}B0 = 3B0 = 30$$

The 7-table

Etc.

The 6-table

$$2 * 6 = 2 * \frac{1}{2}B1 = 1B2 = 12 \text{ TT3, the upper table}$$

The full 6-table

$$6 * 6 = 6 \text{ } 6s = 6 * \frac{1}{2}B1 = 3B6 = 36$$

$$7 * 6 = 6 \text{ } 7s = 6 * \frac{1}{2}B2 = 3B12 = 4B2 = 42$$

$$8 * 6 = 8 \text{ } 6s = 8 * \frac{1}{2}B1 = 4B8 = 48$$

$$9 * 6 = 6 \text{ } 9s = 6 * \frac{1}{2}B4 = 3B24 = 5B4$$

$$\text{ten} * 6 = 6 \text{ tens} = 6B0 = 60.$$

$$\text{Or, } 6 * 6 = (B-4) * (B-4) = BB-8B+16 = 2B16 = 3B6 = 36, \text{ etc.}$$

The full 7-table

$$6 * 7 = 6 \text{ } 7s = 6 * \frac{1}{2}B2 = 3B12 = 4B2 = 42$$

$$7 * 7 = 7 \text{ } 7s = 7 * \frac{1}{2}B2 = 3B(5+14) = 3B19 =$$

$$4B9 = 49$$

$$8 * 7 = 7 \text{ } 7s = 8 * \frac{1}{2}B2 = 4B16 = 5B6 = 56$$

$$9 * 7 = 9 \text{ } 7s = 9 * \frac{1}{2}B2 = 4B(5+18) = 4B23 =$$

$$6B3 = 49$$

$$\text{ten} * 7 = 7 \text{ tens} = 7B0 = 70.$$

$$\text{Or, } 6 * 7 = (B-4) * (B-3) = BB-7B+12 = 3B12 = 4B2 = 42, \text{ etc.}$$

The full 8-table

Etc.

Figure 14. To learn the times table we use our finger to travel on a BundleBundleBoard

- Method 05. (Connected vessels). On our BBBoard we see two vessels next to each other, 6 7s to the left and 0 3s to the right. We then move the top 7s to the right. It is not enough since now we have 5 7s to the left and only 2B1 3 to the right. Moving one more 7s to the right will do the job since now we have 4 7s to the left and 4B2 3s to the right. So, the answer is 4B2 tens or 4ten2 or 42.

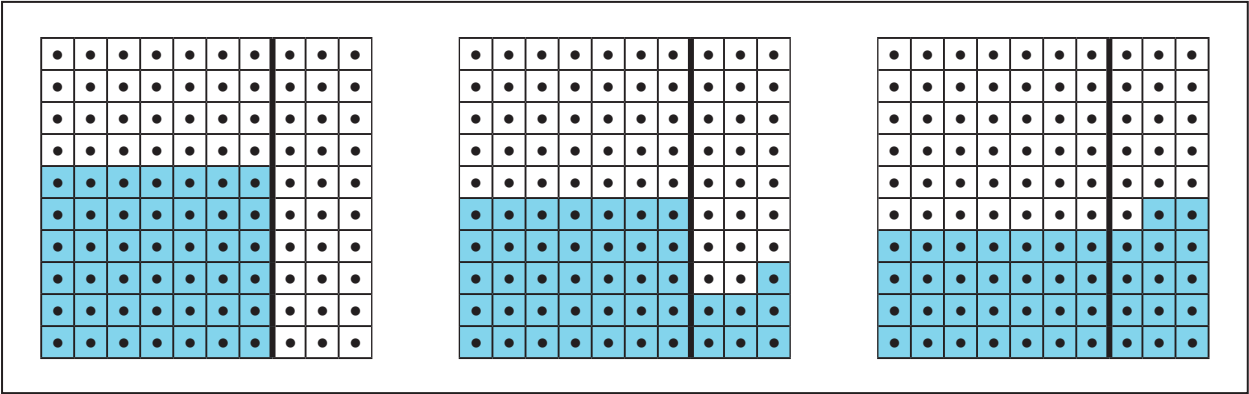


Figure 15. Six 7s recounted to tens by leveling two vessels. The width goes up, the height goes down

F05. Changing from Icons to Icons

Asking “Two 3s is how many 4s?” you see an equation, ‘ $2 * 3 = u * 4$ ’, that you use ‘do the same on both sides to solve as $u = (2 * 3) * 1/4 = 6/4$. We see two 3s recounted in 4s carried out on a BB-Board and tested or predicted by a calculator where entering, $2 * 3/4$. The answer, 1.more’, means ‘1 bundle and some unbundled’ that are found be pulling away the stack, 1 4s, from the 2 3s, which gives 2 unbundled. So, the calculator predicts that 2 3s may be recounted as 1B2 4s, which is also observed on a BBBoard.

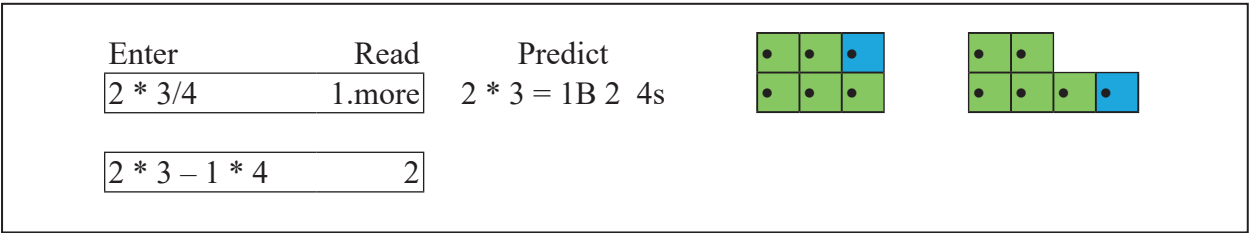


Figure 16. A calculator predicts the result of recounting 2 3s in 4s as 1B2 4s

F06. BundleBundles are squares

On our Bundle-Bundle-Board we see that all the bundle-bundles are squares. And we see that two 2s need two more 2s and one corner to become three 3s, which again need two more 3s and one corner to become four 4s, which again need two more 4s and one corner to become five 5s. So, going up we see that $1BB\ 3s = 1BB\ 2B\ 1\ 2s = 4 + 4 + 1 = 9$, and $1BB\ 4s = 1BB\ 2B\ 1\ 3s = 9 + 6 + 1 = 16$, and $1BB\ 5s = 1BB\ 2B\ 1\ 4s = 16 + 8 + 1 = 25$. So, 1, 4, 9, 16, and 25 are the first squares.

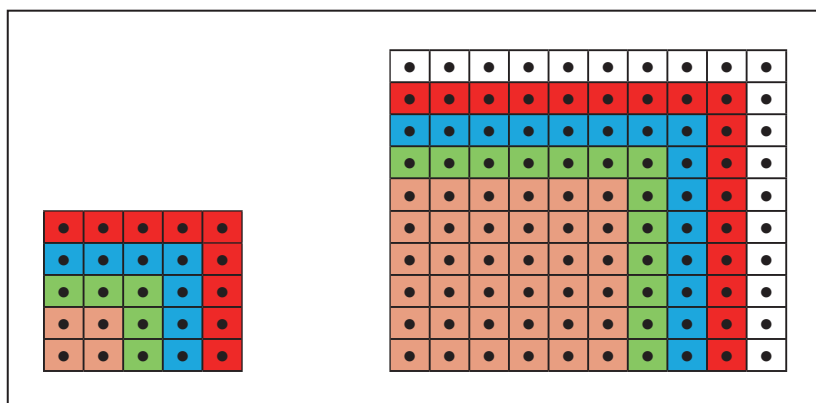


Figure 17. BundleBundles as squares where $1BB - 2B - 1$ or $1BB - 2B - 1$ gives the next or previous

Going down from ten instead, we see that. $1BB - 9s = 1BB - 2B - 1 \text{ tens} = 100 - 20 + 1 = 81$ where 1 is added since it is pulled away twice. And that $1BB - 8s = 1BB - 4B - 4 \text{ tens} = 100 - 40 + 4 = 64$, and that $1BB - 7s = 1BB - 6B - 9 \text{ tens} = 100 - 60 + 9 = 49$, and that $1BB - 6s = 1BB - 8B - 16 \text{ tens} = 100 - 80 + 16 = 36$. The first 9 squares thus are 1, 4, 9, 16, 25, 36, 49, 64, and 81

We see that, of course, 1 and 9 have the same last digit, as has 2 and 8, and 3 and 7, and 4 and 6.

F07. Solving Quadratic Equations

Splitting a BundleBundleBoard in two squares and two stacks solves quadratic equations.

If the sides are split in an unknown number u plus 3 then the board contains a u -square and a 3-square and two $3 * u$ stacks, so if $u^2 + 6 * u + 8$ is zero, then the $(u + 3)$ square only contains $9 - 8 = 1$ which is a 1-square.

This means that u can be -2 or -4 since both $(-2 + 3)$ and $(-4 + 3)$ give 1 when squared.

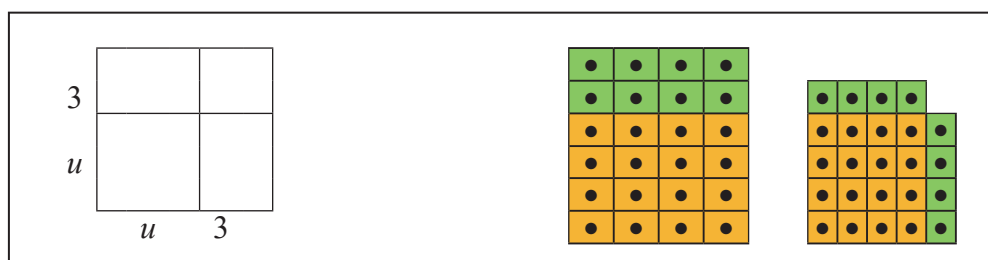


Figure 18. A $u+3$ square solves the equation $u^2 + 6*u + 8 = 0$. A 6 4s stack is being squared

F08. Squaring Stacks

A rectangular stack as 6 4s may be changed into a square by moving half the surplus from the top to the side to make it close to the so-called square root that is the side of the wanted square.

Here the top surplus is $6 - 4 = 2$ 4s, so we move half of 2 4s from the top to the side. But that is too much since we also need to fill the top right corner with a $4 * t$ stack from the side and $t * 4$

stack from the top. This gives $2 * 4 * t = 1$, or $8 * t = 1 = (1/8) * 8$, so $t = 1/8$, so $T = (5 - 1/8) * (5 - 1/8) = 4.88 * 4.88$. Guessing 4.88 as the square root is close to the calculator, $\sqrt{6 * 4} = 4.89...$

F09. Using Per-Numbers when Trading

We see that in shops we can buy apples with money. If the sign says 4\$ per 5kg then we have the per-number 4\$ / 5kg, or 4\$ for each 5kg, or 4/5 \$/kg.

So, if we want to have 20kg we simply recount them in 5s to see how many times we must pay 4\$:

$$20\text{kg} = (20/5) * 5\text{kg} = (20/5) * 4\$ = 16\$.$$

Likewise, 20\$ may be recounted in 4s to see how many times we get 5kg.

$$20\$ = (20/4) * 4\$ = (20/4) * 5\text{kg} = 25\text{kg}$$

F10. Per-Numbers become Fractions with like Units

If my share of a 5\$ bet is 2\$ then from a gain I will get 2\$ per 5\$ or 2\$/5\$ or 2/5. So, we see that per-numbers become fractions with like units. To find 2/5 of 30\$ I simply recount 30 in 5s to see how many times I get 2\$: $30\$ = (30/5) * 5\$$ gives me $(30/5) * 2\$ = 12\$$.

The fraction 2/5 can occur with other units as $(2\text{ }3\text{s}) / (5\text{ }3\text{s}) = (2 * 3) / (5 * 3) = 6/15$. Likewise with the fraction $8/20 = (2 * 4) / (5 * 4) = (2\text{ }4\text{s}) / (5\text{ }4\text{s}) = 2/5$.

My uncle says that this is called to expand or reduce a fraction which we will understand if we later learn about equivalence classes in set products. We don't know what he is talking about so we asked him why we can't understand fractions as special per-numbers. That is because per-numbers do not exist in mathematics, he said. That we really don't understand since per-numbers are the same as bundle-numbers. So maybe they also have no bundle-numbers in mathematics? But then, why shall we learn it, because without bundle-numbers there are no units?

F11. Mutual Recounting in a stack gives Trigonometry

A stack has two internal lines called diagonals. One slopes up, the other down.

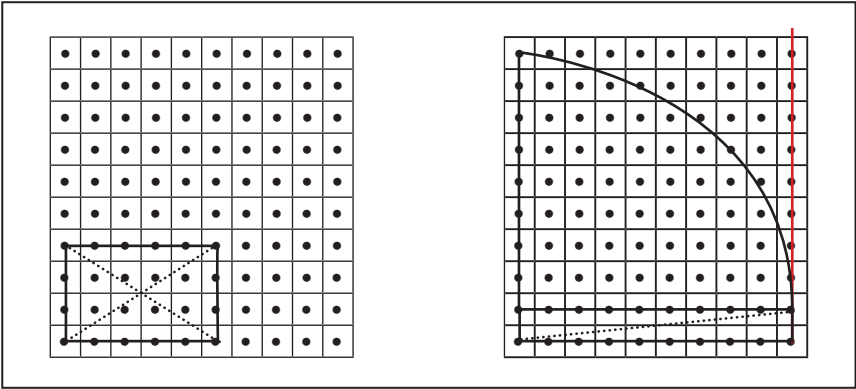


Figure 19. A BBBoard with two internal diagonals and a circle split in small triangles

To find out how steep it is we can recount the up-number in the out-number: $\text{up} = (\text{up/out}) * \text{out}$. Here the per-number, up/out, is called the line's slope or tangent (Angle) number. So, with a 3 per 5 slope, a total of 20 out-meters out will give $(20/5) * 3_{\text{up}} = (20/5) * 3_{\text{up}} = 12$ up-meters.

In a circle with radius 1, half the length is called pi. In the beginning the circle follows its tangent, the line that shows its direction if it was not to hold a constant distance to its center. So here the circle and its tangent are almost of the same length. So, $\pi = n * (\tan(180/n))$ for n large. And my calculator shows that $1000 * \tan(180/1000) = 3.1416$. So, pi must be close to 3.1416.

F12. Triangles on a BundleBundleBoard

Now I will look at totals with other forms than rectangles to see how much they fill. My uncle says that we should use the word area instead of fill-number, so we will do that.

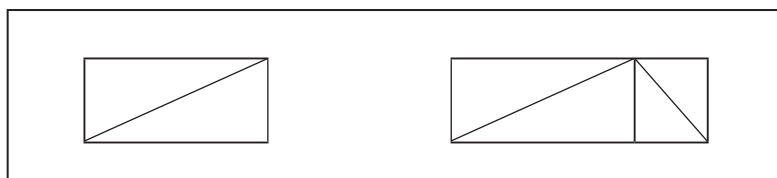


Figure 20. A triangle has half the area of the surrounding stack

First, we see that a diagonal splits a stack in two right triangles with half the area to each. Then we see that any triangle can be split into two right triangles surrounded by a stack. So again, the triangle has half the area of the surrounding stack. Now we look at triangles on a BBBoard.

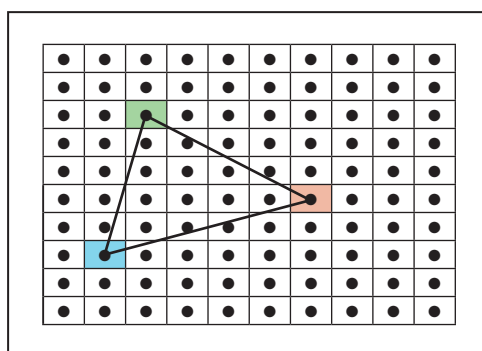


Figure 21. A calculator predicts the result of recounting 2 3s in 4s as 1B2 4s,

We place pearl A on the 2out and 3up (2,3) dot in the upper right corner of a 3 2s stack. Pearl B we place on the (3,8)-dot, and pearl C on the (7,5)-dot. Now we want to find the length and the angles and the area in the triangle ABC that is surrounded by a 5 5s stack.

The AB line is diagonal in a 5 1s stack with a 5/1 diagonal angle. On a calculator, the tangent button tells that this is 79 degrees, so there are $90 - 79 = 11$ degrees to the other angle. We find the length of the diagonal as $\sqrt{26} = 5.1$ by adding the squares of the stack's sides, $5^2 + 1^2 = 26$. My uncle call this the Pythagoras rule. And the area we find as half of $5 * 1$, i.e., $2\frac{1}{2}$.

The BC line is diagonal in a 3 4s stack with a $3/4$ diagonal angle, or 37 degrees with the tangent button, and $90 - 37 = 53$ degrees to the other angle. We find the length of the diagonal as $\sqrt{25} = 5$. And we find the area as half of $3 * 4$, i.e., 6. The AC line is diagonal in a 2 5s stack with a $2/5$ diagonal angle, or 22 degrees with the tangent button, and $90 - 22 = 68$ degrees to the other angle. We find the length of the diagonal as $\sqrt{29} = 5.4$. And we find the area as half of $2 * 5$, i.e., 5.

To find the area of the ABC triangle, we begin with the full area of the surrounding 5 5s stack where $5 * 5 = 25$. Then we pull away the three outer right triangles and get $25 - 5 - 6 - 2\frac{1}{2} = 11\frac{1}{2}$.

To find the angles in the triangle ABC we begin with 90 degrees for A and 180 degrees for B and C. Then we pull away the two neighbor angles in the outer right triangles and get: $A = 90 - 22 - 11 = 57$ degrees, and $B = 180 - 79 - 37 = 64$ degrees, and $C = 180 - 68 - 53 = 59$ degrees. Then, we test the results by adding the three angles: $57 + 64 + 59 = 180$.

Finally, we draw this triangle on a squared paper to test the answers by measuring instead of calculating.

F13. Trips on a BBBoard

Now I will look at my BBBoard, not as a space-board with fixed forms, but as a time-board where I can change place with different trips. I will travel on a line to see when it meets other lines.

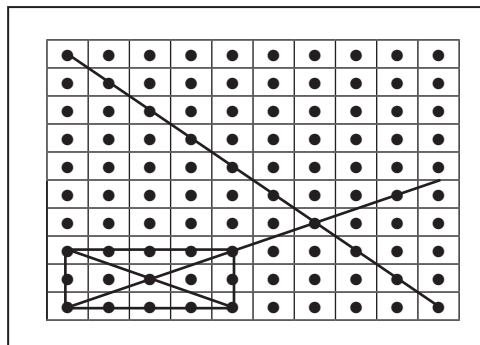


Figure 22. A BBBoard as a time-board where a trip from (0,0) through (4,2) meet another line

On a space-board, the first dot is number 1, but on a time-board it is number 0 since I have not yet changed place. Now, a 2 4s stack will be called a 4x2 box, where 4 is the run- or out-number, and 2 is the rise- or up-number. In a 4x2 box, the two inside diagonals slope up 2 per 4, $2/4$, or down -2 per 4, $-2/4$. Its angle can be found on a calculator to 26.6 degrees up or down.

If we begin at the (0,0)-dot, a 2 per 4, $2/4$, trip will end at the (4,2)-dot. We now continue in the same direction to an unknown (x,y)-dot. The angle hasn't changed so we have that

$$y/x = 2/4, \text{ or } y = 2/4 * x, \text{ or } y = \frac{1}{2} * x.$$

This formula is called the line's equation, and it can find one number if the other is known. If the x-number is 9, then the y-number is

$$y = \frac{1}{2} * x = \frac{1}{2} * 9 = 4.5.$$

And if the y-number is 4, then the x-number is found in the equation $4 = \frac{1}{2} * x$ giving $x = 4 * 2 = 8$.

Another line goes from the (0,9)-dot to the (9,0)-dot. Inside its 9x9 box the diagonal slopes -9/9 or -1/1. So, after x steps y have decreased to $y = 9 - x$. The lines then meet where $y = \frac{1}{2} * x = 9 - x$. Here, we change the unit by recounting x in 2s as $x = (x/2) * 2 = 2 * u$ with $u = x/2$.

Now, $\frac{1}{2} * 2 * u = 9 - 2 * u$, or $u = 9 - 2 * u$, or $u + 2 * u = 9$, or $3 * u = 9$, or $u = 9/3 = 3$.

This gives $u = x/2 = 3$, or $x = 2 * 3 = 6$. Here, $y = 9 - 6 = 3$, so the two lines meet at the (6,3)-dot.

F14. Adding On-top or Next-to

Once counted and recounted, totals may be added on-top or next-to. To add 2 3s and 4 5s on-top as 5s, we must first recount 2 3s to 1B1 5s to get the sum 5B1 5s. To add them as 3s, we must first recount 4 5s to 6B2 3s to get the sum 8B2 3s. Adding 2 3s and 4 5s next-to as 8s we get 3B2 8s by adding the areas, which my uncle calls integral calculus where we multiply before we add.

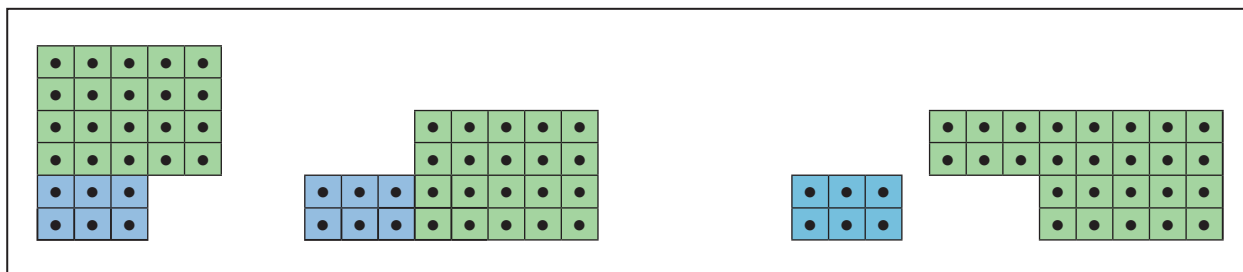


Figure 23. Two 3s and four 5s added on-top or next-to. And the reverse question, $2 \text{ 3s} + ? \text{ 5s} = 4 \text{ 8s}$

Now we ask '2 3s and how many 5s total 4 8s?'. So, we pull away the 2 3s and we recount the rest in 5s. So subtraction comes before division, which my uncle calls differential calculus used to calculate changes as per-numbers. And here the n-square grows with $2 * n$ almost as we saw before.

F15. Adding Squares

To add squares, we take four copies of a 2 3s stack and place them as a 5-square. Inside this square we see two stacks and a 2-square and a 3-square. But we also see a square formed by four diagonals that is surrounded by four half stacks.

So, the two side squares add as the diagonal square. The bottom-top line shows the side of the new square when a 3-square is added to a 4-square.

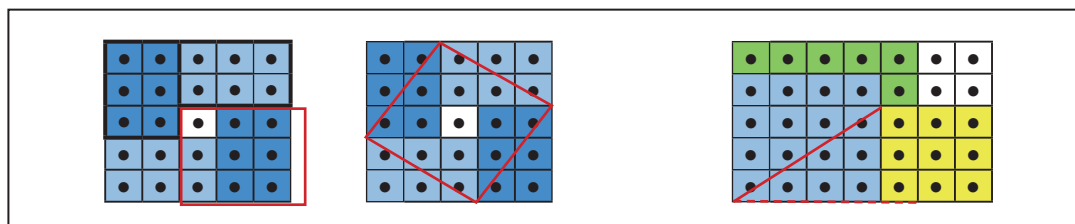


Figure 24. Two squares below and one inside two stacks. Two squares adding as a square

F16. Adding Per-numbers

In a shop we add both kg-numbers and per-numbers in mixtures as 2kg at 3\$/kg plus 4kg at 5\$/kg. We can add the unit-numbers 2 and 4 directly to 6 kg, but before we add the per-numbers they must be multiplied to unit-numbers also, so they add as $2 * 3\$$ and $4 * 5\$$, which is the same as adding 2 3s and 4 5s by using integral calculus.

My uncle says that we later will learn to use integral calculus to add piecewise or locally constant per-numbers as the area under their per-number curve, which is we find by writing the areas as differences so that all the middle terms cancel out when added.

F17. Adding Fractions

Fractions also occur in mixtures as 2 apples with 1/2 red and 3 apples with 2/3 reds. Again, we add the unit numbers directly to 5 apples, whereas the per-numbers must be multiplied first to $\frac{1}{2} * 2$ and $\frac{2}{3} * 3$ that add to 3. So here the sum is 5 apples with 3/5 reds. My uncle says that you will try to teach us that 1/2 plus 2/3 is 7/6, but I think he jokes because there cannot be 7 red among 6 apples.

F18. Adding one-digit Numbers

We see that the textbook wants to teach us that 6+8 is 14. But 6 and 8 cannot exist without units. If we count in 6s, then $6+8 = 1B0 + 1B2 = 2B2$ 6s or 1B4 tens. If we count in 8s, then $6+8 = 1B-2 + 1B0 = 2B-2$ 8s or 1B6 8s or 1B4 tens. If we count in tens we can use half-bundles, $\frac{1}{2}B$, so here $6+8 = 1 \frac{1}{2}B1 + 1 \frac{1}{2}B3 = 1B4$ tens. On a BBBoard we place the 6 and 8 on-top instead of next-to see the sum, and to see that $8-6 = 2$, and that $6-8 = -2$.

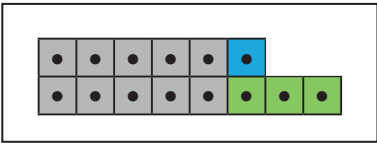


Figure 25. A BBBoard with 6 as 1 ½B 1 on-top of 8 as 1 ½B 3 to see how they add and subtract

F19. Using Underloads and Overloads in Calculations

We don’t need that since bundle-numbers are flexible with both overloads and underloads.

Overload	Underload	Overload	Overload
65	65	7 x 48	336 /7
+ 27	– 27		
6 B 5	6 B 5	7 x 4 B 8	33 B 6 /7
+ 2 B 7	– 2 B 7		
8 B12	4 B-2	28 B 56	28 B 56 /7
9 B 2	3 B 8	33 B 6	4 B 8
92	38	336	48

Figure 26. With Bundles, carrying and place values are unneeded when using under- and overload

On a BBBoard, the two-digit numbers 23 and 47 can be multiplied as 2B 3 and 4B 7 to 10BB 8B 1. And as 10BB 8B 1 we can divide 1081 by 23 as 2B 3 to become 5B -3, or 4B 7 tens.

2B			3		2B			3	
↓	8BB	12B	4B		↑	10BB	15B	? 5B	
↓	14B	21	7		↑	-6B	-9	? -3	
8BB	26B	21			10BB	9B	-9		
10BB	8B	1			10BB	8B	1		

Figure 27. On a BBBoard, $23 * 47 = 1081$, and $1081/23 = 4B7 = 47$

F20. Adding Letters, what is $a + a * b$?

To add, numbers must have like units, so how do we add letters that stand for unknown numbers? Well, all numbers have 1s as a unit, $2 = 2 \text{ 1s} = 2 * 1$, and $a = a \text{ 1s} = a * 1$. And, $a * b$ is really $a \text{ b's}$ with b as the unit, or with a is the unit if we exchange the letters and write $a * b = b * a = b \text{ a's}$.

My uncle calls this the commutative law for multiplication. We see it if turning 2 3s over to 3 2s. It also applies for addition where $2 + 3 = 3 + 2$, but not for subtraction or division where $4 - 2$ and $2 - 4$, and $4/2$ and $2/4$ are not the same.

So, now we can add a and $a * b$ to a total T

$$T = a + a * b = a * 1 + a * b = 1 * a + b * a = 1 \text{ a's} + b \text{ a's} = (1 + b) \text{ a's} = (1 + b) * a.$$

So, we may take a common unit or factor outside a bracket. Or we can do the opposite and write

$$T = a * (1 + b) = a * 1 + a * b = a + a * b$$

Here we take a common outside unit inside a bracket and distribute it to all numbers inside. This my uncle calls the distributive law.

To add, $a * c + a^2 * b$, we choose the common factors as the unit to move outside the bracket.

$$T = a * c + a^2 * b = a * c + a * a * b = c * a + a * b * a = (c + a * b) * a = (c + a * b) \text{ a's}.$$

F21. The Algebra Square Reunites Unlike and Like Unit- and Per-numbers

Counting ten fingers in 3s we get $T = 1BB \text{ 0B } 1 \text{ 3s} = 1 * B^2 + 0 * B + 1$. My uncle calls this an example of a general bundle-formula called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration.

With units we thus see there can be only four ways to unite numbers: addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers.

Going back, subtraction and division split a total into unlike and like unit-numbers. Differentiation splits a total into unlike per-numbers. And finally, a total is split into like factors by the factor-finding root and the factor-counting logarithm. My uncle calls this 'the Algebra square' where the Arabic word Algebra means to reunite.

Calculations unite/ <i>split Totals in</i>	Unlike	Like
Unit-numbers m, s, kg, \$	$T = a + b$ $T - b = a$	$T = a * b$ $\frac{T}{b} = a$
Per-numbers m/s, \$/100\$ = %	$T = \int f \, dx$ $\frac{dT}{dx} = f$	$T = a^b$ $\sqrt[b]{T} = a \quad \log_a T = b$

Figure 28. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

F22. Fact, Fiction and Fake, the three Genres of our Number Language Tales

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers and about totaling and totals in time and space. My uncle calls this modeling. And he says that as in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models.

Fact models talk about the past and present and only need to have the units checked. Fiction models talk about the future and need to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that ‘2+3 = 5’ always despite 2weeks + 3days = 17days, thus transforming mathematics to ‘mathematism’.

My uncle says that if we are allowed to keep and develop our natural bundle-numbers with units then we will all be numerate as asked for by the UN Sustainable Development Goal 4 about quality education. So, allowed to keep and develop our natural intelligence we don’t need any AI. All you must do is to use our Bundle-numbers with units. And no adding the first year, please, only counting and recounting so we really get to know what we are numbering and later calculating. Please, let us Grip to Grasp, at least the first three years? Please, please it is so fun. You will not?

F23. Children’s Declaration of Independence

Well, if you will not use our Bundle-numbers with units we will protect our own number-language and innate number-sense by slightly reformulate the American Declaration of Independence:

“We, the children, declare unanimously that when it becomes necessary for us to dissolve the educational bands which have connected us with you, and to assume among the separate and equal station to which our human nature entitle us, a decent respect requires that we above have declared the causes which impel us to the separation of an education that prohibits this.”

Discussion

We asked the Cinderella question: If mastery of Many is the end goal, essence-math may be a means, but are there differnt means also, e.g., an existence-math, that will make the prince dance? And if so, what is its content and what kind of mathematics does the prince learn during the dance?

We used difference-research building on the observation that when asked “How many years next time?” a 3-year-old child will say “That is not 4, that is 2 2s” if seeing the 4 fingers held 2 by 2. Adapted to the culture, adults only see the essence, four. But still adapting to nature, preschool children only see what exists, bundles of 2s in space and two of them when counted in time. To gather data, we used sociological imagination to visit a grade 1 class in a near decolonized future.

Here we found one content of an existence-math curriculum building on the children bundle-numbers with units as two 3s and four 5s. And we also found that this curriculum allows students to meet the core of mathematics, linearity and calculus, right away when adding their bundle number on-top and next-to as areas.

But first they met power and logarithm when bundle-counting in bundles and bundle-of-bundles. Then they met division, multiplication, subtraction, and addition in that order when using push-away and push-back followed by pull-away and pull-back to recount a total in a specific bundle size with the unbundled placed on-top as decimals, fractions or negative numbers.

Next, they met equations and basic algebra when recounting between tens and icon-units. And with BundleBundle as squares, they also met quadratic equations.

Then, when recounting between physical units, they met per-numbers and fractions, as well as trigonometry that leads on to coordinate geometry before plane geometry. And, since they must be multiplied to unit-numbers before adding, per-numbers and fractions also add next-to as areas thus also using calculus with addition and reversed addition. Adding many differences as one difference here offers a simple way to integrate and to solve difference equations.

So, very soon they created the ‘Algebra Square’ showing how to master Many by uniting and splitting totals into the four kinds of numbers in the world, unlike and like unit-numbers and per-numbers. And with this number-language to number the world for getting numbers to use in forward or reversed calculations, they now can produce quantitative tales having the same genres as has the word language, fact, fiction and fake (Tarp, 2001).

On this direct way to mastery of Many they have learned the core of classical mathematics as well as its essential part so that the rest may be taught as footnotes to those who may find it interesting. At the same time, we can now recommend a curriculum to directly reach mastery of Many as a means to reach the UN Sustainable Development goal that youth and adults all shall be numerate.

A BBM Bundle-Bundle Math Curriculum

By its very nature, multiplicity is multiplicative, not additive. Students therefore should be allowed to keep and develop their innate number sense and their bundle-numbers with units as two 3s where the counting-number in time, two, and the bundle-number in space, 3s, must be treated together.

When counted in 2s, 5 fingers can occur as I I I I I, 0B 5, or as II I I I, 1B 3, or as II II I, 2B 1, or as II II II, 3B -1 2s. But since we bundle in 2s, two bundles is one bundle of bundles, so five can also be IIII I, 1BB 0B 1 2s, which makes ten be 2BB 0B 2, or 1BBB 0BB 1B 0 2s. Here power shows how many 1s there are in a BBB of 2s, $2^3 = 8$. And a logarithm shows how many times bundles have been bundled, $\log_2 8 = 3$.

Counting 8 in 2s means from 8 push-away 2s, iconized by a broom as $8/2$ so the calculator can predict the result, $8/2 = 4$. Here we must tell counting, $8s/2 = 4\ 2s$, from splitting, $8s/2s = 2\ 4s$.

The opposite to ‘push-away’ is ‘push-back-to-stack’ iconized by a lift called multiplication, 4×2 , where the calculator predicts that 4 times stacking 2s gives 8 1s.

Combined push-away and push-back create a 're-count formula' used to change units all over STEEM including economy, $8 = (8/2) * 2$, or with T and B for Total and Bundle, $T = (T/B * B)$.

Pulling-away is iconized by a rope called subtraction, and pulling-back a cross called addition.

Combined pull-away and pull-back create a 're-unite formula' $T = (T - B) + B$.

The two formulas solve equations, which is another word for reversed calculation, by moving 'to opposite side with opposite sign'.

The equation $u * 2 = 8$ is solved by $u = 8/2$ since 8 recounts in 2s as $8 = (8/2) * 2$.

The equation $u + 2 = 8$ is solved by $u = 8 - 2$ since 8 reunites with 2 as $8 = (8 - 2) + 2$.

The equation, $2^u = 8$, is solved by $u = \log_2 8$ since 8 as 2folded gives $8 = 2^3$, and $\log_2 8 = 3$.

The equation, $u^2 = 8$, is solved by $u = \sqrt{8}$ since, by definition, a $\sqrt{8}$ -square has area 8.

Recounting from tens to icons gives equations as $u * 5 = 20$, solved by $u = 20/5$ recounting 20 in 5s.

Recounting from icons to tens gives times tables as 6 7s = ? tens, found by multiplication where $6 * 7 = 6 * \frac{1}{2}B 2 = 3B 12 = 4B 2 = 42$.

Recounting between physical units gives per-numbers as 2\$/3meter that become fractions with like units, $2\$/3\$ = 2/3$, and that bridge the units by recounting:

$$8\$ = (8/2) * 2\$ = (8/2) * 3m = 12m.$$

Recounting the sides in a stack gives trigonometry: $up = (up/out) * out$. Here the per-number, $\tan(\text{Angle}) = up/out$, finds the angle of the diagonal. This allows the length of a half circle to be found as $\pi = n * \tan(180/n)$, for n large.

Two 3s and four 5s add on-top after recounting has made the units like, i or next-to as areas, i.e., as integral calculus where adding many differences as one difference eases integration.

Fractions and per-numbers also add as areas with integral calculus since they must be multiplied to unit numbers before adding.

So, with 2m at 3\$/m and 4m at 5\$/m we get $(2+4)m$ and $(2*3 + 4*5)\$$.

The Algebra Square shows how to master Many by reuniting and splitting a total into the four kinds of numbers in the world, unlike and like unit-numbers and per-numbers.

A number-language to number the world for getting numbers to forward or reverse calculations, may produce quantitative tales with the same genres as in the word-language, fact, fiction and fake.

How Different is the Difference?

Digits now are no longer symbols like letters, but icons with as many sticks as they represent. 3 now is called '1B0 3s' or '0B3 tens'. Ten, eleven and twelve now are also called 'one-bundle-zero', 'one-bundle-one', and 'one-bundle two'. And hundred and thousand are called 'bundle-bundle' and 'bundle-bundle-bundle'.

Multidigit numbers no longer occur without units, $23 = 2B3$). This makes the place value system unneeded. Calculations with overloads and underloads give bundle-numbers with units flexibility that makes carrying and borrowing unneeded, e.g., $46 + 37 = 4B 6 + 3B 7 = 7B 13 = 8B 3 = 83$.

Addition now depends on the units so $2 + 3$ is not 5 by necessity. $2\text{weeks} + 3\text{weeks} = 5\text{weeks}$, but $2\text{weeks} + 3\text{days} = 17\text{days}$. So, without a unit, 3 does not exist, only with a bundle-unit like, e.g., 0B3 tens, or 1B 0 3s, or 1B 1 2s, or 1B -1 4s, or 1B -2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., ' $2 + 3$ ' = (1B 0 + 1B 1) 2s = 2B 1 2s, or ' $2+3$ ' = (1B -1 + 1B 0) 3s = 2B -1 3s = 1B 2 3s. Likewise with subtraction ' $9 - 6$ ' = (1B 3 - 1B 0) 6s = 0B 3 6s = '3', or ' $9 - 6$ ' = (1B0 - 1B-3) 9s = 0B --3 9s = 0B3 9s, showing that minus times minus must be plus.

Also, addition now is not well-defined since two 3s and four 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and $6 * 8$ is not 48 by necessity. Instead, $6 * 8$ exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or in tens: 6 8s is 5B 3 9s, and 4B 8 tens. Division like $8/2$ has two different meanings, $8s / 2s$ gives 4 2s, and $8s / 2 = 2$ 4s.

Solving equations now is different. The equation ' $u * 2 = 8$ ' asks "How many 2s in 8?" which is found by recounting 8 in 2s as $8 = (8/2) * 2$. The solution, $u = 8/2$, now is found by 'moving to opposite side with opposite sign', which follows the formal definition: $8/2$ is the number u that multiplied with 2 gives 8, if $u * 2 = 8$ then $u = 8/2$.

Equations thus no longer are seen as two equivalent numbers-names that remain equivalent if the same operation is performed to both. So, the balancing method now is unneeded with its transformations using the communicative, associative, and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements to addition and multiplication. And we no longer use the neutralizing 'do the same to both sides' lever method to solve the equation $2 * x = 8$ by saying:

$$2 * x = 8 \leftrightarrow (2 * x) * \frac{1}{2} = 8 * \frac{1}{2} \leftrightarrow (x * 2) * \frac{1}{2} = 4 \leftrightarrow x * (2 * \frac{1}{2}) = 4 \leftrightarrow x * 1 = 4 \leftrightarrow x = 4$$

The multiple calculation $2 + 3 * 4$ no longer is 14 by definition or by the 'PEMDAS' rule. With units, $2 + 3 * 4$ exists as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation ' $2 * a + 3 * a = (2+3) * a$ ' no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer 'go over one'. Instead, a per-number like $4\$/5\text{kg}$, links the two units by recounting: $16\$ = (16/4) * 4\$ = (16/4) * 5\text{kg} = 20 \text{ kg}$, and $30\text{kg} = (30/5) * 5\text{kg} = (30/5) * 4\$ = 24\$$.

Fractions no longer are numbers by themselves, instead they are per-numbers with like units, $3\text{meter}/4\text{meter} = \frac{3}{4}$, $3 \text{ meter}/100\text{meter} = 3/100 = 3\%$. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., $1/2 + 2/3 = (1+2)/(2+3) = 3/5$ in this case, and not 7 red of 6 apples.

Trigonometry no longer must wait until after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal. Good news for STEM (Tarp, 2021).

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait until secondary school since Bundle-Bundles are squares that lead directly to the question 'how to square a rectangle' that provides a double split square containing the three parts of a quadratic equation.

The simplicity of the Algebra Square will no longer be hidden. And no longer will models be seen as mere approximations but as tales with three genres, fact and fiction and fake.

And no longer will 8 competences be needed in mathematics education, only the two natural competences, to count and add in time and space (Tarp, 2017).

Teacher education in existence-math could begin by demodeling the concepts in essence-math (Tarp, 2020) and by watching some MrAlTarp YouTube videos, or join the self-instructing online education on the MAHTeCADEMY.net, or read some articles (Tarp, 2016-2025).

So finally, the number-language may have its communicative turn to be learned by telling tales about things and actions in space and time, just as the word-language had around the 1970's as described in H.G. Widdowson's book 'Teaching Language as Communication'.

	Essence-math, mathematism	Existence-math, Many-math
Digits	Symbols	Icons
345	Place value system	$T = 3BB\ 4B\ 5$, $BB = B^2$, $BBB = B^3$
Operations	Functions, order: $+$ $-$ $\times$ $/$ $^$	Icons, opposite order: $^$ $/$ $\times$ $-$ $+$
$3 + 4$	$3 + 4 = 7$	Meaningless without units
$3 * 4$	$3 * 4 = 12$	$3 * 4 = 3\ 4s$, may be recounted to 1.2 tens
$9 = ?\ 2s$	Meaningless, only ten-counting	$9 = 3B3 = 5B-2 = 4B1 = 4\frac{1}{2}\ 2s$
$8 = ?\ 2s$	Meaningless, only ten-counting	$8 = (8/2) * 2$, $T = (T/B) * B$, proportionality
$2 * u = 8$	$2 * u = 8$ then $(2 * u) * \frac{1}{2} = 8 * \frac{1}{2}$ then $(u * 2) * \frac{1}{2} = 4$, then $u * (2 * \frac{1}{2}) = 4$ then $u * 1 = 4$ then $u = 4$	$2 * u = 8 = (8/2) * 2$, so $u = 8/2 = 4$
$6 * 7 = ?$	eh 44? eh 52? eh 42? OK	$6 * 7 = 6 * \frac{1}{2}B\ 2 = 3B\ 12 = 4B\ 2 = 42$ $6 * 7 = (B-4) * (B-3) = (10-4-3) * B + 4 * 3$ $= 3B12 = 4B2 = 42$
$4\ kg = 5\ \$$, $6\ kg = ?$	$1\ kg = 5/4\ \$$, $6\ kg = 6 * 5/4\ \$$	$6\ kg = (6/4) * 4\ kg = (6/4) * 5\ \$$
$1/2 + 2/3 = ?$	$1/2 + 2/3 = 3/6 + 4/6 = 7/6$	$1/2 * 2 + 2/3 * 3 = 3/5 * 5$
$3 + 4 * 5 = ?$	$7 + 5 = 12$, or, $3 + 20 = 27\ ??$	$2 * 3 + 4 * 5 = 3B\ 2\ 8s$ (next-to) or $5B\ 1\ 5s$, or $8B\ 2\ 3s$ (on-top)
$6 + 9 = ?$	$6 + 9 =$ $7, 8, 9, , 10, 11, 12, , 13, 14, \mathbf{15}.$	$2B\ 3\ 6s$ or $2B\ -3\ 9s$ or $\frac{1}{2}B\ 1 + \frac{1}{2}B\ 4 = 1B\ 5\ tens = 15$
$9 - 4$	$9 - 4 = 8, 7, , 6, \mathbf{5}.$	$1B\ 0 - 0B\ 4\ 9s = 0B\ 9 - 0B\ 4 = 0B\ 5\ 9s = 5$ $2B\ 1 - 1B\ 0\ 4s = 1B\ 1\ 4s = 5$ $1\ \frac{1}{2}B\ 4 - (1\ \frac{1}{2}B\ -1) tens = 4 - - 1 = 4 + 1 = 5$
$4 - 9$	Impossible	$1B0 - 2B\ 1\ 4s = 1B0 - 1B\ 5\ 4s = 0B\ -5 = -5$
Tangent = ?	$\tan = \text{sine/cosine}$	$\text{raise} = (\text{raise/run}) * \text{run}$, $\tan = \text{raise/run}$

Figure 29. Overview of the differences between Essence-math and Existence-math

Will the Difference make a Difference, Reactions to a BBM BundleBundle Math Curriculum

At the 9th ICMI-East Asia Regional Conference on Mathematics Education (EARCOME 9) in 2025 in Korea, the Special Sharing Groups accepted my paper “Can a Decolonized Mathematics Secure Numeracy for All?” with the following announcement:

This proposal tackles an urgently needed conversation in mathematics education by challenging deeply ingrained assumptions about number systems and arithmetic instruction and proposing a truly decolonized approach that foregrounds learners’ intuitive “bundle-number” language. Its strength lies in weaving together a compelling theoretical critique—drawing on Habermas’s colonization concept and rich philosophical underpinnings—with concrete instructional innovations like the Algebra Square that reframe operations as intuitive spatial and bundling processes. By aligning this reconceptualization directly with SDG 4’s numeracy targets and illustrating how multiplication-centered reasoning better reflects real-world number use, the paper promises to make a bold and impactful contribution to both research and practice. We look forward to seeing how this work can reshape numeracy instruction and foster truly inclusive mathematical literacy for all.

In Topic Study Groups, reviewers asked for empirical studies where existence- and essence-math had been compared. My answer was that testing both at the same time is ethically problematic:

Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order: addition, subtraction, multiplication, division, power. And only few will hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. And, since counting and recounting precedes adding, they will not meet addition until they meet the core of math directly, calculus and linearity. What will happen if a class B student changes to class A, or the other way around? In short, how ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return far ahead to normal education? Ethical testing is for teacher education and home education.

Conclusion and Recommendations

We now can answer the Cinderella question: If mastery of Many is the end goal, essence-math may be a means for a few, but there is a different means also, an existence-math, which builds upon counting and adding in time and space where counting and recounting to change units come before adding, and which leads to the core of essence-math allowing the rest to be taught as footnotes.

The recommendation therefore is to teach existence-math at least the first three years, and to postpone addition to the second year to allow the children to have fun with counting and recounting the outside world while using a BundleBundleBoard to represent their numbers and a calculator to predict their actions.

Presenting statistically significant testing results would be desirable, or would it because of the ethical challenges in comparing a ‘flat-earth’ and a ‘round-earth’ curriculum? As a difference researcher you can hope to find a difference, but you cannot always hope to test it to see if the difference makes a difference. However, in order to test it we first must have a difference to test. So, maybe it is best to let the composer compose, and then leave it to others to play the music?

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